Modular forms of half integral weights, noncongruence subgroups, metaplectic groups

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Abstract

The lecture notes are based on the number theory topics course on 3 Feb, 2016.

1 modular forms of half integral weights

Let $\Gamma \subset SL2(Z)$ be a finite index subgroup. Let k be an integer. Recall a weight k, level Γ modualr form is a holomorphic function on the upper half plane satisfying the funcitonal equation: $f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau)$ for $\gamma \in \Gamma$

Definition 1.1. *Half integral weight modular forms are holomorphic functions on the upper half plane with the modified functional equation: $f(\gamma\tau) = \epsilon(\gamma)(c\tau+d)^(k/2)f(\tau)$ for $\gamma \in \Gamma$ where ϵ is some root of unity and the square root is chosen in some half plane.*

Example 1.2. $\theta(\tau) = \sum exp(2\pi i n^2\tau)$

$\Gamma(8)$=*congruence subgroup mod 8, then* $\theta(\gamma(\tau)) = \begin{cases} \theta(\tau) & c = 0 \\ (\frac{c}{d})(c\tau+d)^{1/2}\theta(\tau) & c > 0 \end{cases}$
where $(\frac{c}{d})$ *is the Legendre symbol.*

Exercise 1.3. *For all N, there exist $\gamma \in \Gamma(N)$, such that the Legende symbol $(\frac{c}{d}) = -1$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$*

For integral weight forms the transformation law is simple: $j(\gamma,\tau) = (c\tau+d)^k$ then $j(\gamma_1\gamma_2,\tau) = j(\gamma_1,\gamma_2\tau)j(\gamma_2,\tau)$ so $j(\gamma,\tau)$ is a multiplier system.

But $(c\tau+d)^1/2$ is not a multiplier system.

2 The metaplectic group

Definition 2.1. $Mp_2(\mathbb{R}) = \{(g,\phi) | g \in SL2(\mathbb{R}), \phi : H \mapsto \mathbb{C}, \phi^2 = c\tau+d\}$

We see $Mp_2(\mathbb{R})$ has a natural covering map to $SL2(\mathbb{R})$. $Mp_2(\mathbb{R})$ is a Lie group but not the real points of an algebraic group; in particular it cannot be realised by a matrix representation.

The group law is given by:

$(g,\phi) * (g\prime,\phi\prime) = (gg\prime, \tau \mapsto \phi(g\prime\tau)\phi\prime(\tau))$

Recall the θ function satisfies some functional equation. This means the factor of automorphy forms a multiplier system. This fact is equivalent to:

The covering map $Mp_2(\mathbb{R}) \mapsto SL_2(\mathbb{R})$ splits on $\Gamma(8)$ with the splitting given by $(\frac{c}{d})(c\tau + d)^{1/2}$

Remark 2.2. *The way to prove this is indeed a multiplier system: either use the fact that the theta function is nonzero, or use quadratic reciprocity.*

3 Congruence subgroup problem for SL_n

Question: if $\Gamma \subset SL(O_K)$ has finite index, where K is a number field, is Γ a congruence subgroup?

Here the congruence subgroup means the coefficients of the matrix equals the identity matrix mod the ideal (n).

Example 3.1. *For $SL_2(Z)$, the answer is no.*

Take $\Gamma \subset SL_2(\mathbb{Z})$ small enough so that Γ is not torsion free. Then Γ is a free group, so there is a surjection $\Gamma \mapsto \mathbb{Z}$.

Let $\hat{\Gamma} = \varprojlim \Gamma/\Upsilon$ Υ has finite index in Γ.

Let $\bar{\Gamma} = \varprojlim \Gamma/\Gamma(n)$.

The hom from Γ to $\mathbb{Z}$ extends to $\hat{\Gamma} \mapsto \hat{\mathbb{Z}}$.

$\bar{\Gamma}$ is the closure of Γ in $SL_2(\mathbb{A}_f)$.

Since SL_2 is semisimple, the commutator map is surjective, $[sl_2, sl_2] \mapsto sl_2$.

So $[\bar{\Gamma}, \bar{\Gamma}]$ is open in $SL_2(\mathbb{A}_f)$, since $\bar{\Gamma}$ is open in $SL_2(\mathbb{A}_f)$. So $[\bar{\Gamma}, \bar{\Gamma}]$ has finite index in $\bar{\Gamma}$.

Hence there is no hom $\bar{\Gamma} \mapsto \hat{\mathbb{Z}}$ apart from 0.

There is $1 \mapsto C \mapsto \hat{\Gamma} \mapsto \bar{\Gamma} \mapsto 1$.

C is called the congruence kernel.

Theorem 3.2. *The theorem of Bass-Milnor-Serre says that if n is greater or equal to 3, and the number field K has a real place, then every subgroup of finite index in $SL_n(O_K)$ is a congruence subgroup.*

If K is totally complex there will be a noncongruence subgroup.

Let K be totally complex, and contains an n-th root of unity. We can define the n-th power Legendre symbol on K, as follows:

Let $a \in K$, p=prime ideal in O_K, p does not divide na, then

$a^{\frac{Np-1}{n}}$=some n-th root of unity mod p.

Define the Legendre symbol $(\frac{a}{p})$ to be the n-th root of 1.

For a general ideal coprime to na, define the Legendre symbol by the product law.

Define $\Gamma(n^2)$ to be the congruence subgroup in $SL_2(O_K)$ mod the ideal (n^2).

Define a map $\kappa : \Gamma(n^2) \mapsto \mu_n$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{cases} (\frac{c}{d}) & c \neq 0 \\ 1 & c = 0 \end{cases}$$

Theorem 3.3. *Kubata: κ is a hom, and its kernel is a noncongruence subgroup.*

Exercise 3.4. *Prove this.*

Bass-Milnor-Serre extended the κ to $SL_m(O_K, n^2)$.

κ gives an isomorphism between the congruence kernel and μ_n as long as n is the total number of roots of unity in K.

This means every subgroup of finite index in $SL_m(O_K, n^2)$ contains some $\Gamma(N) \cap ker(\kappa)$. (If either m is at least 3 or [K:$\mathbb{Q}$]) is at least 4).

Remark 3.5. *Kubata's exercise is equivalent to the reciprocity formula for the Legendre symbol in K, ie the Artin reciprocity law for Kummer extensions of K.*

4 Digression on K theory

Before going on, define the K2 group of a field. Let K be any infinite field. The group $SL_m(K)$ is perfect for m at least 3, meaning it is equal to its own commutator subgroup.

Hence $SL_m(K)$ has a universal central extension.

$1 \mapsto K2(K) \mapsto St_m(K) \mapsto SL_m(K) \mapsto 1$

Here K2(K) is defined to be the kernel. It does not depend on m as long as m is at least 3.

We recall what it means to be a universal central extension: for any Abelian group A, the central extensions of the form

$1 \mapsto A \mapsto ? \mapsto SL_m(K) \mapsto 1$

are in bijective correspondence with the hom set

$Hom(K2(K), A)$

where the correspondence is given by the obvious morphism of extension sequences.

For a field K, the group K2(K) is calculated by Matsumoto as follows (giving a presentaion of K2(K)):

$K2(K) = K^* \otimes_{\mathbb{Z}} K^* / < a \otimes 1 - a, a \in K \setminus \{0, 1\} >$

We will write $\{a, b\}$ for the image of the tensor $a \otimes b$ in K2(K).

Remark 4.1. *In terms of matrices this means:*

$[\widetilde{diag(a, a^{-1}, 1, \ldots, 1)}, \widetilde{diag(b, b^{-1}, 1, \ldots, 1)}] \in K_2(K)$

*Notice we need at least 3*3 matrices for this to make sense. The ˜means taking the preimage in $St_m(K)$.*

We also get an extension sequence for SL_2:

$1 \mapsto K2(K) \mapsto something \mapsto SL_2(K) \mapsto 1$

by taking the middle term to be the preimage of $SL_2(K)$ in $St_3(K)$.

This extension is easy to describe: here is a inhomogeneous 2-cocycle.

$\sigma(g, h) = \{X(gh)/X(g), X(gh)/X(h)\}, g, h \in SL_2(K)$

$X(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{cases} c & c \neq 0 \\ d & c = 0 \end{cases}$

This satisfies the cocycle relation.

$\sigma(g_1 g_2, g_3)\sigma(g_1, g_2) = \sigma(g_1, g_2 g_3)\sigma(g2, g3)$

Remark 4.2. *The cocycle condition is equivalent to the associativity of the group law on $SL_2(K) \times K_2(K)$.*

Exercise 4.3. *Show σ is a 2-cocycle. (Need properties of $\{a, b\}$): the bilinearity of the tensor and the relation $\{x, 1 - x\}) = 1$ for $x \neq 1$.*

5 Hilbert symbol, metaplectic group again

Let $\mathbb{Q}_p$=either a p-adic field or the real numbers. Define for $a, b \in \mathbb{Q}_p$

$(a,b)_p = \begin{cases} 1 & ax^2+by^2=1 \text{has a solution in } \mathbb{Q}_p \\ -1 & \text{if not} \end{cases}$

For the real number case,

$(a,b) = \begin{cases} 1 & a>0 \text{ or } b>0 \\ -1 & a,b<0 \end{cases}$

The (a,b) is called the Hilbert symbol and it satisfies the bilinear relations and the property that $(x, 1-x) = 1$ for $x \neq 1$.

In other words the Hilbert symbol is a hom $K_2(\mathbb{Q}_p) \mapsto \{1, -1\}$. In fact it is the only nontrivial such.

For the real number case we get a central extension of $SL_2(\mathbb{R})$ which reproduces our $Mp_2(\mathbb{R})$. This is a unique connected double cover.

Note: if G=Lie group, then G is homotopic to the maximal compact subgroup. In the case of $SL_2(\mathbb{R})$, the maximal compact subgroup is the circle, so the first fundamental group is $\mathbb{Z}$, hence there is a unique connected double cover.

The quadratic reciprocity can be stated as:

$a, b \in \mathbb{Q}^*, \prod_{\text{p prime or infinity}} (a,b)_p = 1$

For each prime we have a central extension

$1 \mapsto \mu_2 \mapsto \widetilde{SL_2(\mathbb{Q}_p)} \mapsto SL_2(\mathbb{Q}_p) \mapsto 1$

defined by the relavent two-cycle σ_p.

We can put these together to obtain an adelic version:

$1 \mapsto \mu_2 \mapsto \widetilde{SL_2(\mathbb{A})} \mapsto SL_2(\mathbb{A}) \mapsto 1$

where $\sigma_{\mathbb{A}} = \prod \sigma'_p$, and σ'_p is cohomologous to σ_p.

By the Hilbert symbol version of the reciprocity law, the cocycle $\sigma_{\mathbb{A}}$ splits on $SL_2(\mathbb{Q})$.

It turns out if p is odd, then σ_p splits on $SL_2(\mathbb{Z}_p)$ and σ_2 splits on $SL_2(\mathbb{Z}_2, 4)$. $\sigma_{\mathbb{A}}$ will split on $U = \prod_{podd} SL_2(\mathbb{Z}_p) \times SL_2(\mathbb{Z}_2, 4)$.

Now on $\Gamma(4)$ we have two different splittings of almost the same extension (the difference between the two extensions is σ_∞).

If we divide one splitting by another, we get a map $\kappa : \Gamma(4) \mapsto \mu_2$. If these were two different splittings of the same cocycle, κ would be a hom. But if they are not, then κ is a splitting of σ_∞), ie, $\sigma_\infty(g,h) = \kappa(g)\kappa(h)/\kappa(gh)$.

Remark 5.1. *This is how we show $\kappa(\gamma)(c\tau+d)^{1/2}$ is a multiplier system. And when we work out what κ is, we get $\kappa(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (\frac{c}{d})$*

Example 5.2. *If K is totally complex, then*

$SL_2(K_\infty) = SL_2(\mathbb{C})^N, K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R}$

$SL_2(\mathbb{C})$ is simply connected, ie, it has no nontrivial covering groups. Complex Hilbert symbols are 1.

So the extension

$1 \mapsto \mu_n \mapsto \widetilde{SL_2(\mathbb{A})} \mapsto SL_2(\mathbb{A}) \mapsto 1$

splits on $SL_2(K)$ by reciprocity law, and also splits on $U \times SL_2(K_\infty)$.

$\Gamma(n^2) = SL_2(K) \cap (U \times SL_2(K_\infty))$.

On $\Gamma(n^2)$ we have two splittings of the same extension.

Dividing one extension by another, we get a hom $\kappa : \Gamma(n^2) \mapsto \mu_n$.

This is exactly the same κ we had before. $ker(\kappa)$ is a noncongruence subgroup.

Remark 5.3. *metaplectic forms are automorphic forms on $G(\hat{\mathbb{A}})$ for any reductive G over a number field.*