Nested Poisson

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1 Nested Poisson PMF

The PMF of a Poisson random variable (denoted $Po(\lambda)$) with constant rate λ is as follows:

$$P(X = k|X \sim Po(\lambda)) = \frac{e^{-\lambda}\lambda^k}{k!} \tag{1}$$

We can see that the PMF of a Poisson random variable generated from a rate proportional by a constant μ to a Poisson random variable with a rate λ is equal to the following due to the law of total probability:

$$P(X = k|X \sim Po(\mu Po(\lambda))) = \sum_{\tau=0}^{\infty} \frac{e^{-\mu\tau}(\mu\tau)^k}{k!} \cdot \frac{e^{-\lambda}\lambda^\tau}{\tau!} \tag{2}$$

Setting the constant $e^{-\mu}\lambda = \alpha$ we see:

$$\sum_{\tau=0}^{\infty} \frac{e^{-\mu\tau}(\mu\tau)^k}{k!} \cdot \frac{e^{-\lambda}\lambda^\tau}{\tau!} = \sum_{\tau=0}^{\infty} \frac{e^{-\lambda}(\mu\tau)^k}{k!} \cdot \frac{\alpha^\tau}{\tau!} = \frac{e^{-\lambda}\mu^k}{k!} \sum_{\tau=0}^{\infty} \frac{\tau^k\alpha^\tau}{\tau!} \tag{3}$$

As we know that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ we can then show Touchards k^{th} polynomial is of use in reducing the equation:

$$\frac{e^{-\lambda}\mu^k}{k!} \sum_{\tau=0}^{\infty} \frac{\tau^k\alpha^\tau}{\tau!} = \frac{e^{\alpha-\lambda}\mu^k}{k!} T_k(\alpha) \tag{4}$$

Where $T_k(\alpha)$ is Touchard's k^{th} polynomial. Thus:

$$P(X = k|X \sim Po(\mu Po(\lambda))) = \frac{e^{\alpha-\lambda}\mu^k}{k!} T_k(\alpha) \tag{5}$$