Newton's Method Cycles

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Abstract

Based on the paper *Sometimes Newton's Method Cycles* [2], we first asked ourselves if there were any Newtonian Method Cycle functions which have non-trivial guesses. We encountered a way to create functions that cycle between a set number of points with any initial, non-trivial guesses when Newton's Method is applied. We exercised these possibilities through the methods of 2-cycles, 3-cycles and 4-cycles. We then generalized these cycles into k-cycles. After generalizing Newton's Method, we found the conditions that skew the cycles into a spiral pattern which will either converge, diverge or become a near-cycle. Once we obtained all this information, we explored additional questions that rose up from our initial exploration of Newton's Method.

1   Introduction

1.1   Newton's Method

Newton's Method[1], also known as the Newton-Raphson method, is a root-finding algorithm that works by getting successively better approximations to the roots of real-valued functions. The method in one variable is used as follows:

The method starts with a function f defined over the real numbers x, the function's derivative f', and an initial guess x_0 for a root of the function f. If the function satisfies the assumptions made in the derivation of the formula and the initial guess is close, then the manner in which Zaytman [4] presents a better approximation for x_1 is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Geometrically, $(x_1, 0)$ is the intersection with the x-axis of the tangent to the graph of f at $(x_0, f(x_0))$. The process that Zaytman [4] presented is repeated as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until a sufficiently accurate value is reached.

1.2   History

Newton's Method is derived from Isaac Newton's description of a special case of the method in "On Analysis by Equations with an infinite number of terms" (written in 1669 and published in 1671 by William Jones)[6]. Newton applied the method only to polynomials. He does not compute the successive approximations x_n, but computes a sequence of polynomials, and only at the end arrives at an approximation for the root x. Newton viewed the method as purely algebraic and makes no mention of the connection with calculus. Newton's method was first published in 1685 in *A Treatise of Algebra both Historical and Practical* by John Wallis. In 1690, Joseph Raphson published a simplified description in *Analysis aequationum universalis*. Raphson again viewed Newton's method purely as an algebraic method and restricted its use to polynomials, but he describes the method in terms of the successive approximations x_n instead of the more complicated sequence of polynomials used by Newton. Finally, in 1740, Thomas Simpson described Newton's method as an iterative method for solving general nonlinear equations using calculus, essentially giving us the use described above.

1.3   How Newton's Method Behaves

Before we use Newton's Method, we must understand what outcomes we should get from using this formula. There are two ways that Newton's Method behaves – converge and

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[1]This function is meant to be used in iteration.

diverge. When a function converges, the roots of the function will be found for the unknown variable. There are, however, different ways in which the method can fail:

1. If the first derivative is not well behaved in the neighborhood of a particular root, the method may overshoot and diverge from that root.

2. If a critical point of a function is encountered, the derivative will be zero and the method cannot be used because of division by zero.

3. If there is a large error in the initial estimate then it may contribute to non-convergence of the algorithm.

4. The algorithm cycles between a set number of points, never reaching the root.

In this paper we will focus on how Newton's Method fails by cycling. There are also many different ways for Newton's Method to cycle, such as 2-cycle, 3-cycle, 4-cycle. This pattern continues for any k-cycle.

To use the method, we start with an initial guess which is reasonably close to the true root, then the function is approximated by its tangent line and one computes the x-intercept of this tangent line, which is easily done with elementary algebra. This x-intercept will typically be a better approximation to the function's root than the original guess, and the method can then be iterated.
As stated above, one of the ways that Newton's Method behaves is by converging. Where the formula continually finds a more accurate number of the true root through iteration. Here is an example from a video [3] demonstrating when Newton's Method converges.

$$x^2 - 4 = 2x - 3$$

$$x^2 - 2x - 1 = 0$$

$$h(x) = x^2 - 2x - 1$$

$$h'(x) = 2x - 2$$

$$\text{True root } x = 1 - \sqrt{2} \approx -0.4142136$$

$$x_1 = 0$$

$$x_2 = 0 - \frac{0^2 - 2 \cdot 0 - 1}{2(0) - 2}$$

$$= -\frac{-1}{-2}$$

$$= -\frac{1}{2}$$

$$x_3 = -\frac{1}{2} - \frac{(-\frac{1}{2})^2 - 2(-\frac{1}{2}) - 1}{2(-\frac{1}{2}) - 2}$$

$$= -\frac{1}{2} - \frac{\frac{1}{4} + 1 - 1}{-1 - 2}$$

$$= -\frac{1}{2} - \frac{\frac{1}{4}}{-3}$$

$$= -\frac{6}{12} - \frac{1}{-12}$$

$$= -\frac{5}{12}$$

$$\approx -0.41667$$

The second way that Newton's Method behaves is in the instance that a function does not converge. We will focus on non-convergence by cycling and the ways in which Newton's Method can cycle are very similar, but they all have one difference: they all cycle between the number of points that is in the name. For example, 2-cycle has two points it cycles between, x_0 and x_1, then repeats in a loop using only these two points. For 3-cycle and 4-cycle, they do the same thing as 2-cycles, but 3-cycle loops between 3 points (x_0, x_1, x_2) and 4 points (x_0, x_1, x_2, x_3) for 4-cycles. k-cycles loops between k points $(x_0, x_1, \ldots, x_{k-1})$. k is an undetermined number of cycles. Here is an example of a 2-cycle.

$$f(x) = x^3 - 2x + 2, \qquad x_0 = 0$$

$$x_1 = 0 - \frac{0^3 - 2(0) + 2}{3(0)^2 - 2}$$

$$= 1$$

$$x_2 = 1 - \frac{1^3 - 2(1) + 2}{3(1)^2 - 2}$$

$$= 0$$

1.4 Dynamical Systems

Unlike iterative processes where time is measured in discrete intervals such as years or generations, differential equations are examples of continuous dynamical systems wherein time is a continuous variable [1]. Unlike functions that converge that do not change, dynamics is the study of change. Dynamical system is how a system of variables interacts and changes with time. It can be used to answer questions about the economy, the stock market, simplified climate models, reactive or radioactive chemicals in groundwater, etc. In general, the rates of change will depend on the values of the other variables and this is what makes dynamical systems interesting. For the most part, applications fall into three broad categories: predictive (also referred to as generative), in which the objective is to predict future states of the system from observations of the past and present states of the system, diagnostic, in

which the objective is to infer what possible past states of the system might have led to the present state of the system (or observations leading up to the present state), and, finally, applications in which the objective is neither to predict the future nor explain the past but rather to provide a theory for the physical phenomena. These three categories correspond roughly to the need to predict, explain, and understand physical phenomena.

2 Newton's Method Cycling

2.1 2-cycles

In this paper, a method for finding a 2-cycle with symmetry was employed on simplified discreet dynamical systems. For higher order cycles, they found that complex powers of x are needed and that all functions cycle symmetrically.

Consider applying Newton's Method to $f(x) = x^p$

$$x_{n+1} = x_n - \frac{x_n{}^p}{p \cdot x_n{}^{p-1}}$$

$$= \left(1 - \frac{1}{p}\right) x_n$$

To find 2-cycles we use $p = \frac{1}{2}$ to obtain

$$x_{n+1} = -x_n$$

This will give us a function that will alternate any initial guess with its negative counterpart. Given this condition and applying it to Newton's Method, here we solve for an equation that guarantees a symmetrical 2-cycle:

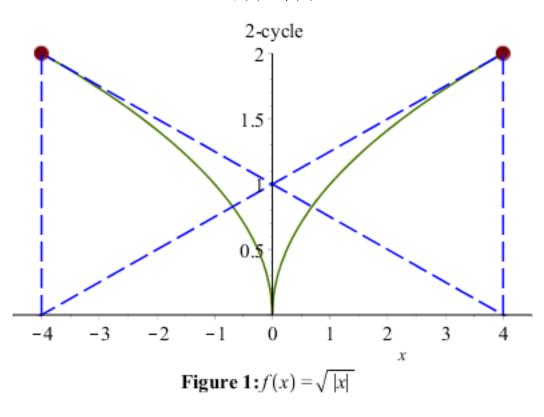$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -x_n$$

Replace any $f(x_n)$ with y and x_n with x to create a differential equation that can be easily solved

$$x - \frac{y}{y'} = -x$$

$$\frac{y}{y'} = 2x$$

$$y' = \frac{y}{2x}$$

$$\int \frac{1}{y} dy = \int \frac{1}{2x} dx$$

$$\ln(y) = \frac{1}{2}\ln(|x|) + c$$

$$y = \sqrt{e^{\ln(|x|)}} \cdot e^c$$

$$y = c \cdot \sqrt{|x|}$$

for $x > 0$ for $x < 0$

$$y = c \cdot \sqrt{x}$$ $$y = c \cdot \sqrt{-x}$$

Since the constant c in the solution does not affect the symmetry of the function, then by setting $c = 1$ we obtain the following 2-cycle which will cycle for any initial guess $x_0 \neq 0$

$$f(x) = \sqrt{|x|}$$

Figure 1: $f(x) = \sqrt{|x|}$

2.2 3-cycles

To find 3-cycles, we will assume $f(x) = x^p$ and apply Newton's Method to obtain

$$x_{n+3} = \left(1 - \frac{1}{p}\right) x_{n+2}$$
$$= \left(1 - \frac{1}{p}\right)^2 x_{n+1}$$
$$= \left(1 - \frac{1}{p}\right)^3 x_n$$

Since this is a 3-cycle and x_{n+2} is the third point in the 3-cycle, we make $x_{n+3} = x_n$ thus creating a repeating cycle. Since we know that $x_{n+3} = x_n$, then

$$x_{n+3} = \left(1 - \frac{1}{p}\right)^3 x_n$$

can simplify to

$$1 = \left(1 - \frac{1}{p}\right)^3$$

Solving for p, we find

$$p = \frac{1}{1 - \sqrt[3]{1}}$$

Using the root 1 does not yield an answer we can use, but using the other two true cubic roots of one, $\left(-\frac{1}{2} \pm \frac{i*\sqrt{3}}{2}\right)$, we find that

$$p = \frac{1}{1 - \sqrt[3]{1}} = \frac{1}{\frac{3}{2} \pm \frac{\sqrt{3}i}{2}}$$

Based on the solution from the paper *Sometimes Newton's Method Always Cycles* [2], they found that when they used the other roots of p,

$$p = \frac{3 \pm \sqrt{3}i}{6}$$

Which, when simplified, come out to be the exact same.

Since we have found p, our initial condition of $f(x) = x^p$ can now be converted into a 3-cycle[2]:
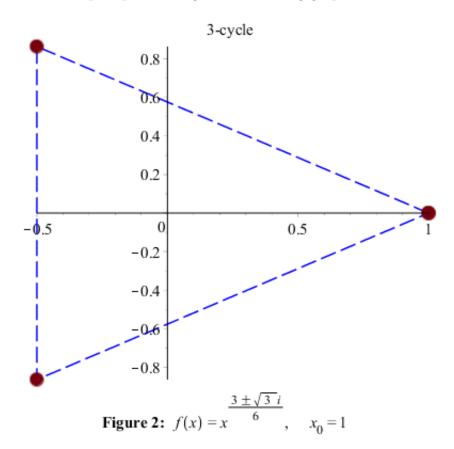
$$f(x) = x^{\frac{3 \pm \sqrt{3}i}{6}}$$

Now let us consider the function $f(x)$ that we have just found and apply Newton's Method three times to it to verify that we cycle back to the starting point. Let's take a starting

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[2]This will go into the complex plane since this is raised to a complex number.

value, $x_0 = 1$.

$$x_1 = 1 - \frac{1^{\frac{3+\sqrt{3}i}{6}}}{\left(\frac{3+\sqrt{3}i}{6}\right) \cdot 1^{\frac{3+\sqrt{3}i}{6}-1}}$$

$$= -\frac{1}{2} + \frac{\sqrt{3}i}{2}$$

$$x_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right) - \frac{\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{\frac{3+\sqrt{3}i}{6}}}{\left(\frac{3+\sqrt{3}i}{6}\right) \cdot \left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{\frac{3+\sqrt{3}i}{6}-1}}$$

$$= -\frac{1}{2} - \frac{\sqrt{3}i}{2}$$

$$x_3 = \left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right) - \frac{\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)^{\frac{3+\sqrt{3}i}{6}}}{\left(\frac{3+\sqrt{3}i}{6}\right) \cdot \left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)^{\frac{3+\sqrt{3}i}{6}-1}}$$

$$= 1$$

When plotted in the complex plane, we get the following graph:

3-cycle

Figure 2: $f(x) = x^{\frac{3 \pm \sqrt{3}\,i}{6}}$, $x_0 = 1$

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2.3   4-cycles

In order to find a function that will give us a 4-cycle we will assume $f(x) = x^p$ and use the same techniques used when finding the 3-cycle function to obtain

$$
\begin{aligned}
x_{n+4} &= \left(1 - \frac{1}{p}\right) x_{n+3} \\
&= \left(1 - \frac{1}{p}\right)^2 x_{n+2} \\
&= \left(1 - \frac{1}{p}\right)^3 x_{n+1} \\
&= \left(1 - \frac{1}{p}\right)^4 x_n
\end{aligned}
$$

This can be simplified to

$$
1 = \left(1 - \frac{1}{p}\right)^4
$$

Solving for p, we find

$$
p = \frac{1}{1 - \sqrt[4]{1}}
$$

Here we find that the roots ± 1 will not give us an answer we can use because $+1$ will give us an undefined p and -1 will give us $p = \frac{1}{2}$, which does not cycle, but using the other two fourth roots of one, $\pm i$, we obtain

$$
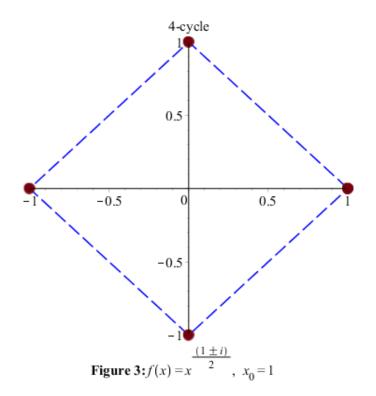p = \frac{1}{1 - \sqrt[4]{1}} = \frac{1}{1 \pm i} = \frac{1 \pm i}{2}
$$

Having found p, we can now obtain our 4-cycle function

$$
f(x) = x^{\frac{1 \pm i}{2}}
$$

Now let us verify that this function does cycle between four points by setting the starting value $x_0 = 1$.

$$x_1 = 1 - \frac{1^{\frac{1+i}{2}}}{\left(\frac{1+i}{2}\right) \cdot 1^{\frac{1+i}{2}-1}}$$

$$= i$$

$$x_2 = i - \frac{i^{\frac{1+i}{2}}}{\left(\frac{1+i}{2}\right) \cdot i^{\frac{1+i}{2}-1}}$$

$$= -1$$

$$x_3 = (-1) - \frac{(-1)^{\frac{1+i}{2}}}{\left(\frac{1+i}{2}\right) \cdot (-1)^{\frac{1+i}{2}-1}}$$

$$= -i$$

$$x_4 = (-i) - \frac{(-i)^{\frac{1+i}{2}}}{\left(\frac{1+i}{2}\right) \cdot (-i)^{\frac{1+i}{2}-1}}$$

$$= 1$$

When plotted in the complex plane, we get the following graph:

Figure 3: $f(x) = x^{\frac{(1 \pm i)}{2}}$, $x_0 = 1$

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2.4 k-cycles

Jumping ahead to the undetermined amount of cycles, k-cycles, we obtain

$$x_{n+k} = \left(1 - \frac{1}{p}\right)^k \cdot x_n$$

This equation shows that if we put any arbitrary number in for x_{n+k} or x_n that this will not affect the outcome and the below equation. Based on algebra, we can divide by the same numbers leaving us with 1.

$$1 = \left(1 - \frac{1}{p}\right)^k \cdot 1$$

Using and implementing the same axiom theorems, steps, and techniques we used with 3-cycles and 4-cycles for k-cycles, we obtain

$$1 = \left(1 - \frac{1}{p}\right)^k$$

Solving for p we obtain

$$p = \frac{1}{1 - \sqrt[k]{1}}$$

For any k, we find that if the roots are ± 1. They are unusable because $+1$ will give us an undefined p and -1 will give us $p = \frac{1}{2}$, which does not cycle, but we can use the other k^{th} roots of 1 to solve for p.

There is another way to find what p is without using Newton's formula. First, we must go back to the formula we used for k-cycles since the formula is similar to this formula.

$$1 = \left(1 - \frac{1}{p}\right)^k$$

Since we are using the generic version of this formula, we must change a couple parts about this formula and even simplify this formula.

$$x_{n+k} = \left(1 - \frac{1}{p}\right)^k \cdot x_n$$

$$x_n \cdot x_k = \left(1 - \frac{1}{p}\right)^k \cdot x_n$$

$$x_k = \left(1 - \frac{1}{p}\right)^k$$

From the paper Sometimes Newton's Method Always Cycles [2], we can note that p can be stated in a different way.

$$p = a + (b \cdot i)$$

There are three properties about our new generic formula.

1. Converge

$$\left|1 - \tfrac{1}{p}\right| < 1$$

$$\left|1 - \tfrac{1}{a+(b\cdot i)}\right| < 1$$

2. Diverge

$$\left|1 - \tfrac{1}{p}\right| > 1$$

$$\left|1 - \tfrac{1}{a+(b\cdot i)}\right| > 1$$

3. Constant

$$\left|1 - \tfrac{1}{p}\right| = 1$$

$$\left|1 - \tfrac{1}{a+(b\cdot i)}\right| = 1$$

To help us further understand what is happening in this formula, we must solve for a.

$$1 = \left|1 - \frac{1}{a + (b \cdot i)}\right|$$

$$1 = \left|1 - \frac{1}{(a + bi)}\frac{(a - bi)}{(a - bi)}\right|$$

$$1 = \left|1 - \frac{a - bi}{a^2 + b^2}\right|$$

$$1 = \left|\frac{a^2 + b^2 - a + bi}{a^2 + b^2}\right|$$

$$1 = \left|\frac{a^2 + b^2 - a}{a^2 + b^2} + \frac{bi}{a^2 + b^2}\right|$$

$$1 = \sqrt{\frac{(a^2 + b^2 - a)^2}{(a^2 + b^2)^2} + \frac{b^2}{(a^2 + b^2)^2}}$$

$$1 = \frac{a^4 + 2a^2b^2 - 2a^3 - 2ab^2 + a^2 + b^4 + b^2}{a^4 + 2a^2b^2 + b^4}$$

$$a^4 + 2a^2b^2 + b^4 = a^4 + 2a^2b^2 - 2a^3 - 2ab^2 + a^2 + b^4 + b^2$$

$$0 = -2a^3 - 2ab^2 + a^2 + b^2$$

$$2a^3 + 2ab^2 = a^2 + b^2$$

$$2a(a^2 + b^2) = a^2 + b^2$$

$$2a = \frac{a^2 + b^2}{a^2 + b^2}$$

$$2a = 1$$

$$a = \frac{1}{2}$$

In the paper, Sometimes Newton's Method Always Cycles [2], the authors mention the function $f(x) = x^p$ can be rewritten as

$$f(x) = x^{a+(b \cdot i)}$$

When we did the calculations for this formula, we found that $f(x)$ also can be

$$f(x) = x^{a-(b \cdot i)}$$

Since the function cycles, having the signs change is arbitrary. As long as the initial choice is constant throughout the rest of the calculations. So then, all we have left is to find b.

Now that we know $a = \frac{1}{2}$, we can obtain the value for b in the function $f(x) = x^{\frac{1}{2}+(b \cdot i)}$.

$$
\begin{aligned}
x_1 &= 1 - \frac{1}{\frac{1}{2} + (b \cdot i)} \\
&= 1 - \frac{\frac{1}{2} - bi}{\frac{1}{4} + b^2} \\
&= \frac{\frac{1}{4} + b^2}{\frac{1}{4} + b^2} - \frac{\frac{1}{2} - bi}{\frac{1}{4} + b^2} \\
&= \frac{\frac{1}{4} + b^2 - \frac{1}{2} + bi}{\frac{1}{4} + b^2} \\
&= \frac{1 + 4b^2 - 2 + 4bi}{1 + 4b^2} \\
&= \frac{4b^2 - 1}{4b^2 + 1} + \frac{4b}{4b^2 + 1}i
\end{aligned}
$$

Since b is now the only variable we do not know, we need to find it to make sure that the function is sound. But before we do that, we need to understand the cycle itself. For this, we need to look at an example of a cycle.
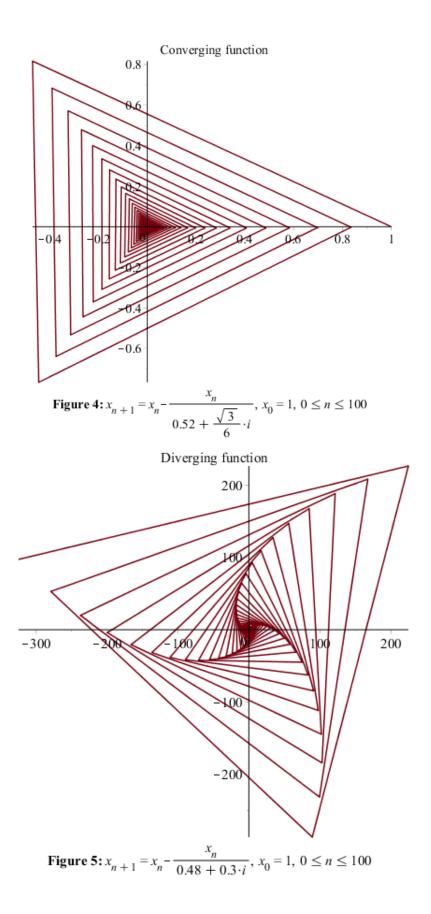
The figures below are examples of converging and diverging cycles. What makes a function converge or diverge is the value of a.
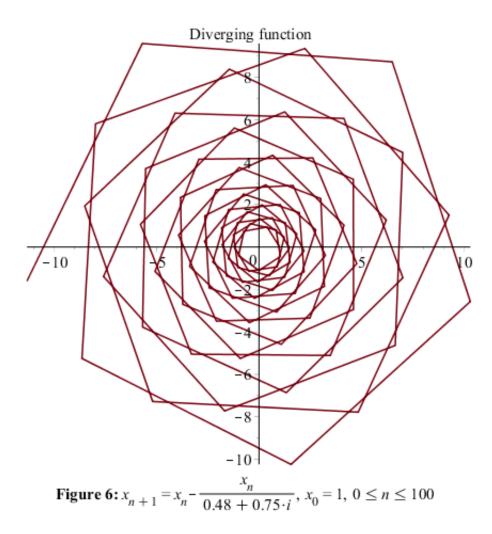
1. Converge

   $a > \frac{1}{2}$

2. Diverge

   $a < \frac{1}{2}$

Converting function

$$\text{Figure 4: } x_{n+1} = x_n - \frac{x_n}{0.52 + \frac{\sqrt{3}}{6} \cdot i}, \ x_0 = 1, \ 0 \leq n \leq 100$$

Diverging function

$$\text{Figure 5: } x_{n+1} = x_n - \frac{x_n}{0.48 + 0.3 \cdot i}, \ x_0 = 1, \ 0 \leq n \leq 100$$

Diverging function

Figure 6: $x_{n+1} = x_n - \dfrac{x_n}{0.48 + 0.75 \cdot i}$, $x_0 = 1$, $0 \le n \le 100$

We first must break the cycle down into more basic parts. Since each iteration is a point with two components, a real and an imaginary part, then we can treat them as right triangles and use trigonometry to find $\cos(\alpha)$. In this case the real part of the equation will be the adjacent side and the imaginary part will be the opposite side of the triangle, so then we have to calculate the hypotenuse of this triangle.

$$
\begin{aligned}
h &= \sqrt{\left(\frac{4b^2 - 1}{4b^2 + 1}\right)^2 + \left(\frac{4b}{4b^2 + 1}\right)^2} \\
&= \sqrt{\frac{16b^4 - 8b^2 + 1 + 16b^2}{16b^4 + 8b^2 + 1}} \\
&= \sqrt{\frac{16b^4 + 8b^2 + 1}{16b^4 + 8b^2 + 1}} \\
&= 1
\end{aligned}
$$

Now that we know what the hypotenuse is, we can now find what b is. The variable α will be useful to help find b because α is a known variable since it is the angle that is between the hypotenuse and the adjacent side of the triangle that was created earlier. We can express

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this through the following equation:

$$\cos(\alpha) = \frac{\left(\frac{4b^2-1}{4b^2+1}\right)}{1} = \frac{4b^2-1}{4b^2+1}$$

When we look at the cycle as a whole, we must note that for each point within the cycle, we can see that there are subsections of the cycle, which are created by the points within the cycle. These points are extremely meaningful in trying to understand what the value of b is. For this, we must introduce one new variable and reintroduce a known variable, k and m. k is the number of points in the cycle, thus the meaning of 2- cycle, 3-cycle, and k-cycle. m is the length between points in the cycle, which is the subsection of cycle we mentioned earlier. Since a cycle, no matter the form, can be summarized as a circle, we can state that a full cycle has a value of 2π. Now knowing this, we now understand what α is.

$$\alpha k = 2\pi$$
$$\alpha = \frac{2\pi}{k}$$

Now to solve for b which is the imaginary component.

$$\cos\left(\frac{2\pi m}{k}\right) = \frac{4b^2-1}{4b^2+1}$$
$$\cos(\alpha) = \frac{4b^2-1}{4b^2+1}$$
$$(4b^2+1)(\cos(\alpha)) = (4b^2-1)$$
$$(4b^2(\cos(\alpha)) + \cos(\alpha) = 4b^2-1$$
$$-4b^2 + (4b^2(\cos(\alpha)) + \cos(\alpha) = -1$$
$$4b^2(-1+\cos(\alpha)) + \cos(\alpha) = -1$$
$$4b^2(-1+\cos(\alpha)) = -1 - \cos(\alpha)$$
$$4b^2 = \frac{-1-\cos(\alpha)}{-1+\cos(\alpha)}$$
$$b^2 = \frac{1}{4} \cdot \frac{-1-\cos(\alpha)}{-1+\cos(\alpha)}$$
$$b = \sqrt{\frac{1}{4} \cdot \frac{-1-\cos(\alpha)}{-1+\cos(\alpha)}}$$
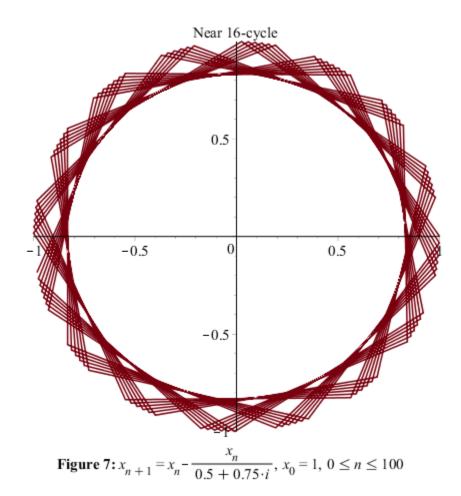$$b = \pm\frac{1}{2} \cdot \sqrt{\frac{-1-\cos(\alpha)}{-1+\cos(\alpha)}}$$
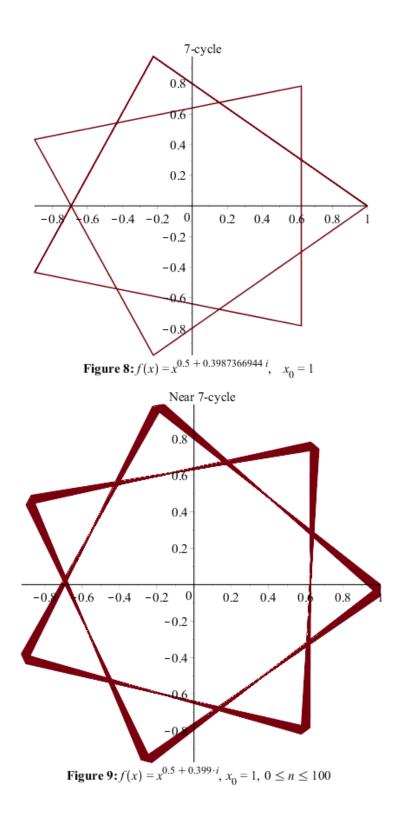$$b = \pm\frac{1}{2} \cdot \sqrt{\frac{1+\cos(\alpha)}{1-\cos(\alpha)}}$$
$$b = \pm\frac{1}{2} \cdot \cot(\alpha)$$

Now from this now we can obtain a general function $f(x) = x^{\frac{1}{2}\pm\frac{1}{2}\cot(\frac{2\pi m}{k})i}$, where k can be input in to create any k-cycle.

2.5   Near Cycles

These are near cycles which do not cycle perfectly between a k number of points. Instead they group in the general area where the specific point is supposed to be. They are generated by keeping $a = \frac{1}{2}$ and b does not satisfy $b = \pm\frac{1}{2} \cdot \cot(\frac{2\pi m}{k})i$. This creates a slightly faster (or slower), but more inaccurate answer for each iteration. These graphs are examples of near cycles.

Figure 7: $x_{n+1} = x_n - \dfrac{x_n}{0.5 + 0.75 \cdot i}$, $x_0 = 1$, $0 \le n \le 100$

Figure 8: $f(x) = x^{0.5 + 0.3987366944\,i}, \quad x_0 = 1$

Figure 9: $f(x) = x^{0.5 + 0.399 \cdot i}, \; x_0 = 1, \; 0 \leq n \leq 100$

3 Other Questions

After our investigation on Newton's Method cycling, we began thinking about if there are other functions aside from x^p that would behave as a cycle. After many searches, we found

a function that does not cycle, but instead it diverges for any initial guess when applying Newton's Method. We first start with the Newton Method with a function $T(x)$ such that

$$\frac{f'(x)}{f(x)} = \frac{T(x) + xT'(x)}{xT(x)}$$

Upon replacing this in the Newton Method we obtain

$$x_1 = x_0 - \frac{f(x)}{f'(x)}$$

$$= x_0 - \frac{x_0 T(x_0)}{T(x_0) + x_0 \cdot T'(x_0)}$$

$$= x_0 \left(1 - \frac{T(x_0)}{T(x_0) + x_0 \cdot T'(x_0)}\right)$$

In order to obtain $T(x)$ we note that

$$\frac{T(x) + x \cdot T(x)}{x \cdot T(x)} = (\ln(x \cdot T(x)))'$$

Let us call $y = f(x)$ and combining this with our first condition we obtain

$$\frac{y'}{y} = (\ln(x \cdot T(x)))'$$

$$\frac{dy}{dx} \cdot \frac{1}{y} = (\ln(x \cdot T(x)))'$$

$$\int \frac{1}{y} dy = \int (\ln(x \cdot T(x)))' dx$$

$$\ln(|y|) = \ln(x \cdot T(x)) + c$$

$$y = x \cdot T(x) \cdot e^c, \quad e^c = D$$

$$f(x) = D \cdot x \cdot T(x)$$

For the quantity $\frac{T(x_0)}{T(x_0) + x_0 \cdot T'(x_0)}$, we must create a condition. We will set this quantity equal to k. We will also let $D = 1$.

$$k = \frac{T(x_0)}{T(x_0) + x_0 \cdot T'(x_0)}$$

$$T(x_0) = k \cdot T(x_0) + k \cdot x_0 \cdot T'(x_0)$$

$$k \cdot x_0 \cdot T'(x_0) = T(x_0) - k \cdot T(x_0)$$

$$T'(x_0) = \frac{T(x_0) - k \cdot T(x_0)}{k \cdot x_0}$$

Now we will replace the $T'(x_0)$ with $\frac{dT}{dx}$, all the $T(x_0)$'s with T, and all the x_0 with x.

$$\frac{dT}{dx} = \frac{T - k \cdot T}{k \cdot x}$$

$$\frac{dT}{dx} = \frac{T \cdot (1 - k)}{k \cdot x}$$

$$\int \frac{1}{T} dT = \frac{1 - k}{k} \int \frac{1}{x} dx$$

$$\ln(T) = \frac{1 - k}{k} \cdot \ln(x) + c$$

$$T = \left(e^{\ln(x)}\right)^{\frac{1-k}{k}} \cdot e^c, \quad e^c = H$$

$$T = H \cdot x^{\frac{1-k}{k}}$$

Therefore our solution is $T(x) = H \cdot x^{\frac{1-k}{k}}$. Putting this back into our equation for $f(x)$ we obtain $f(x) = D \cdot x \cdot T(x) = D \cdot x \cdot H \cdot x^{\frac{1-k}{k}}$. We can simplify this to $f(x) = A \cdot x^{\frac{1}{k}}$. This is a general method to find functions $f(x)$ where all points diverge and that are different from the ones mentioned within the paper Sometimes Newton's Method Always Cycles [2]. For these functions, $k > 2$ will still be of the form x^p, just that p will be none of the values that the paper Sometimes Newton's Method Always Cycles, so the function above will be a different and new function similar to the form x^p.

Since the functions that we have looked at throughout this paper are symmetric, we questioned if there are functions that cycle asymmetrically. After researching this question, we found that there are functions that do cycle asymmetrically [5]. But this requires PhD level mathematics that we currently do not have.

Another question we asked ourselves was what would happen if Newton's Method was multiplied by a constant, h. In other words, how would a weighted Newton's Method cycle? This is what deduced from multiplying the method by h. For this proof, it must be noted that $x_{n+1} = x_n$.

$$x_{n+1} = x_n + \frac{h \cdot f(x_n)}{f'(x_n{}^p)}$$

$$= x_n + \frac{h \cdot x_n{}^p}{p \cdot x_n{}^{p-1}}$$

$$= x_n + \frac{h \cdot x_n}{p}$$

$$= \left(1 + \frac{h}{p}\right)^k \cdot x_n$$

$$1 = \left(1 + \frac{h}{p}\right)^k$$

Here we find that $p = \frac{h}{1 - \sqrt[k]{1}} = h(a + bi)$. Here are a couple of examples comparing the original cycle with its weighted counterpart:

3-cycle, h = 1 (original 3-cyle)

$$x_0 = 1$$

$$x_1 = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$$

$$x_2 = -\frac{1}{2} - \frac{\sqrt{3}i}{2}$$

$$x_3 = 1$$

3-cycle, h = 2 (weighted)

$$x_0 = 1$$

$$x_1 = 1 - \frac{3}{3 - \sqrt{3}i}$$

$$x_2 = \frac{\sqrt{3}i - 1}{8}$$

$$x_3 = -\frac{1}{8}$$

4-cycle, h = 1 (original 4-cycle)

$$x_0 = 1$$

$$x_1 = i$$

$$x_2 = -1$$

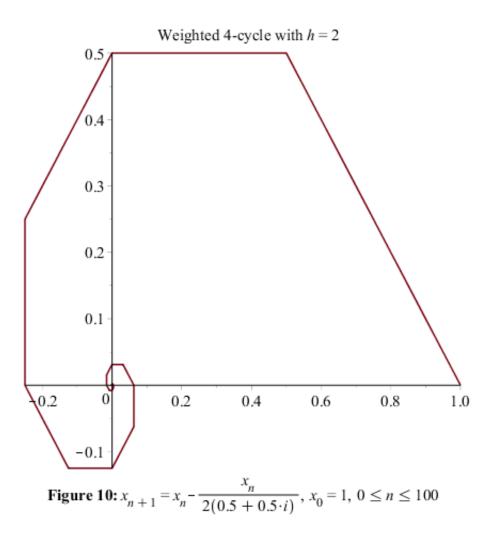$$x_3 = -i$$

$$x_4 = 1$$

4-cycle, h = 2 (weighted)

$$x_0 = 1$$

$$x_1 = \frac{1}{2} + \frac{i}{2}$$

$$x_2 = \frac{i}{2}$$

$$x_3 = -\frac{1}{4} + \frac{i}{4}$$

$$x_4 = -\frac{1}{4}$$

Figure 10: $x_{n+1} = x_n - \dfrac{x_n}{2(0.5 + 0.5 \cdot i)}$, $x_0 = 1$, $0 \le n \le 100$

After completing a couple of calculations for 3 and 4 cycles with different values of h, we found that multiplying Newton's Method by any constant would disrupt the cycle to the point that the cycle would no longer be a cycle, but rather, a converging or diverging function depending on the value of h. In fact, we found it interesting that the function continued to act like a cycle in the case where $h = 2$, in the sense that when the cycle came back to the same line as the original point, but at a smaller coordinate. It appears to be making a inward spiral to the center, therefore converging to 0.

We found the behavior for $h > 1$ converges and for $h < 1$ diverges. We noted that the type of cycle also changes as h changes since it multiplies bi which is what determines the type of cycle we have. We also found that the shape of the graph was congruent every time that the points crossed the x-axis.

After this, we wondered if there was a way to create a weighted cycle with a constant

radius, so we used the same techniques from our generic formula.

$$1 = \left| 1 - \frac{h}{a + (b \cdot i)} \right|$$

$$1 = \left| 1 - \frac{h}{(a + bi)} \frac{(a - bi)}{(a - bi)} \right|$$

$$1 = \left| 1 - \frac{ha - hbi}{a^2 + b^2} \right|$$

$$1 = \left| \frac{a^2 + b^2 - ha + hbi}{a^2 + b^2} \right|$$

$$1 = \left| \frac{a^2 + b^2 - ha}{a^2 + b^2} + \frac{hbi}{a^2 + b^2} \right|$$

$$1 = \sqrt{\frac{(a^2 + b^2 - ha)^2}{(a^2 + b^2)^2} + \frac{h^2 b^2}{(a^2 + b^2)^2}}$$

$$1 = \frac{a^4 + 2a^2 b^2 - 2ha^3 - 2hab^2 + h^2 a^2 + b^4 + h^2 b^2}{a^4 + 2a^2 b^2 + b^4}$$

$$a^4 + 2a^2 b^2 + b^4 = a^4 + 2a^2 b^2 - 2ha^3 - 2hab^2 + h^2 a^2 + b^4 + h^2 b^2$$

$$0 = -2ha^3 - 2hab^2 + h^2 a^2 + h^2 b^2$$

$$2ha^3 + 2hab^2 = h^2 a^2 + h^2 b^2$$

$$2ha(a^2 + b^2) = h^2(a^2 + b^2)$$

$$2ha = \frac{h^2(a^2 + b^2)}{a^2 + b^2}$$

$$2ha = h^2$$

$$a = \frac{h}{2}$$

Since $a = \frac{h}{2}$, then we find that the weighted function will still converge or diverge depending on the value of h since $a \neq \frac{1}{2}$. The only way for this condition to be met would be when $h = 1$, but that would be an unweighted function.

4 Conclusion

With this new found knowledge, we can conclude that Newton's Method does cycle for functions with any initial non-trivial guesses. We have looked into this for all cases and proving for the first three types of cycles, 2-cycles, 3-cycles and 4-cycles. After reaching the conclusion that there are functions that do iterate, we moved forward to the generic function of k-cycles, where we found that functions exist for any number of cycles to satisfy Newton's Method. Proving this, we then became curious about the behaviors of these functions in terms of convergence, divergence or having a constant radius. This brought forth a new idea, near cycles. We discovered that there are functions that behave similarly to cycles, but do not actually cycle perfectly. After our discoveries about these ideas, we began to raise other, more outlying, questions. From these questions, we found another type of function that cycles and is not in the form $f(x) = x^p$, that there is a way for the method to cycle asymmetrically, and what happens when we multiply Newton's Method cycles by a constant.

Newton's Method always cycles!

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