SOLUTION OF $x^x = 2$

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Consider the function $f(x) = x\exp(x)$, defined on the interval $[-1, \infty)$. This is a continuous function, and it is increasing since

$$f'(x) = (x+1)\exp(x) > 0$$

for all $x > -1$. Therefore, f is invertible.

The inverse of f is known as Lambert's W function, and it satisfies

$$W(t)\exp(W(t)) = t$$

for all $t \geq -1/e$. This equation can be rewritten as

$$W(t) = \frac{t}{\exp(W(t))} \tag{1}$$

We will use Lambert's W function to solve the equation $x^x = 2$. Note that the equation has exactly one solution in the positive reals by the Intermediate Value Theorem, since x^x increases continuously from 1 to 4 as x increases from 1 to 2.

$$\begin{aligned}
x^x &= 2 && \text{Given} \\
\ln(x^x) &= \ln(2) && \text{Apply ln to both sides} \\
x\ln(x) &= \ln(2) && \text{Simplify} \\
t\exp(t) &= \ln(2) && \text{Substitute } t = \ln(x) \\
t &= W(\ln(2)) && \text{Definition of } W \\
x &= \exp(W(\ln(2))) && \text{Apply exp to both sides} \\
x &= \ln(2)/W(\ln(2)) && \text{By equation (1)} \\
x &\approx 1.55961046946.
\end{aligned}$$

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