When Area and Perimeter are "Equal"

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Abstract

Various geometrical shapes are described, for which the numerical value of the perimeter is the same as that of the area. Cases of one or two parameters are explored.

1 Introduction

While doing basic geometry problems regarding area and perimeter, I have found situations where the numerical value of the area of a figure and its perimeter are the same. That led me to explore the possible conditions when perimeter and area give the same numerical answer. Of course, the units of measure are different (length vs. area).

A few notes of introduction:

- In this discussion, mention will occasionally be made of integer solutions.
- Since the area and perimeter have the same value, that number will be denoted by K.

2 Cases with only one parameter

2.1 The circle

The relevant equation for a circle of radius R is:

$$\pi R^2 = 2\pi R \tag{1}$$

The solutions are $R = 2$ or 0. Zero produces an unusual geometry, and we will not consider the zero answer for all further analysis. For $R = 2$, area and perimeter $K = 4\pi \approx 12.566$. So area and perimeter will be equal when the diameter is 4. The number 4 will appear in many different settings in this discussion.

2.2 The square

In a similar fashion, for a square with side S:

$$S^2 = 4S \tag{2}$$

then $S = 4, K = 16$.

2.3 The semicircle

Again with radius R:

$$\frac{1}{2}\pi R^2 = \pi R + 2R \tag{3}$$

The solution is $R = 2 + 4/\pi \approx 3.273, K = 2\pi + 8 + 8/\pi \approx 16.83$.

2.4 The equilateral triangle

If the length of each side of the equilateral triangle is S, the height is $\frac{\sqrt{3}}{2}S$:

$$\frac{1}{2}S\left(\frac{\sqrt{3}}{2}S\right) = 3S \tag{4}$$

So then, $S = 4\sqrt{3} \approx 6.928$, the height is equal to 6, and $K = 12\sqrt{3} \approx 20.785$.

Note the presence of the number 4, and also the tangent of the angle, since $\tan(\pi/3) = \sqrt{3}$.

2.5 The isosceles right triangle

For an isosceles right triangle with leg length a:

$$\frac{1}{2}a^2 = \left(2 + \sqrt{2}\right) a \tag{5}$$

So then $a = 4 + 2\sqrt{2} \approx 6.828$, the hypotenuse is $4 + 4\sqrt{2} \approx 9.657$, and $K = 12 + 8\sqrt{2} \approx 23.314$.

Note the presence of the number 4, and also of $\tan(\pi/4) = \sqrt{2}$.

2.6 The 30-60-90 triangle

With the short side of length a, longer side b, hypotenuse c:

$$\frac{\sqrt{3}}{2}a^2 = \left(3 + \sqrt{3}\right) a \tag{6}$$

The solution, side lengths, area and perimeter follow:

$$\begin{aligned} a &= 2 + 2\sqrt{3} \approx 5.464 \\ b &= \sqrt{3}a = 6 + 2\sqrt{3} \approx 9.464 \\ c &= 2a = 4 + 4\sqrt{3} \approx 10.928 \\ K &= 12 + 8\sqrt{3} \approx 25.856 \end{aligned}$$

Note the presence of the number 4, and also $\tan(\pi/3) = \sqrt{3}$.

2.7 The general regular polygon

For completeness, we include the general formula for a regular polygon, though it is not often included in elementary geometry curricula.

Consider a regular polygon, with n as the number of sides, and side length x. The line segment drawn from the center of a regular polygon which is perpendicular to a side is called the apothem of the regular polygon. The apothem is the perpendicular bisector of a side of the polygon. The right triangle formed by the apothem and half the side contains the angle adjacent to the apothem, with measure π/n.

We will show the derivation of the formula for area of the regular polygon determined by side length x, with a as the length of the apothem, and then make the connection to the area and perimeter relationship:

$$\tan(\pi/n) = \frac{x/2}{a} \tag{7}$$

$$a = \frac{x}{2\tan(\pi/n)} \tag{8}$$

$$\text{Area of polygon} = (2n)\,(\text{area of right triangle}) \tag{9}$$

$$= (2n)\left(\frac{1}{2}\left(\frac{x}{2}\right)\frac{x}{2\tan(\pi/n)}\right) \tag{10}$$

$$= \frac{nx^2}{4\tan(\pi/n)} \tag{11}$$

So then, setting area equal to perimeter:

$$\frac{nx^2}{4\tan(\pi/n)} = nx \tag{12}$$

$$x = 4\tan(\pi/n) \tag{13}$$

Replacing this value of x in the equation for the apothem (Equation 8) yields a value of $a = 2$ for all regular polygons where area equals perimeter. This was previously seen in the square of sides equal to 4, and the circle of diameter 4, since a circle can be described as a regular polygon with an infinite number of sides. In fact, the circle with diameter 4 will be the circle inscribed in any regular polygon of this type, and the area of that circle will equal its perimeter.

Table 1 shows values for some of these polygons. Note that the solutions are approximate, except for the square.

As the number of sides increases, the area and perimeter become close to that of the circle ($4\pi \approx 12.566$), while the length of a single side approaches zero.

Other items of interest: a regular polygon with 13 sides of length 1 has an area of approximately 13.2 square units. A regular polygon with 10 sides of length 1.3 has an area of approximately 13.003 square units.

Figure 1 shows the shapes which have been considered so far.

Table 1: Regular polygon where area equals perimeter

Number of sides	**Length of each side**	K
3	6.928	20.78461
4	4	16
5	2.906	14.53085
6	2.309	13.85641
7	1.926	13.48409
8	1.657	13.25483
9	1.456	13.10293
10	1.300	12.99679
11	1.175	12.91957
12	1.072	12.86156
13	0.986	12.81685
14	0.913	12.78163
15	0.850	12.75339
16	0.796	12.73039
17	0.748	12.7114
18	0.705	12.69554
19	0.667	12.68216
20	0.634	12.67076

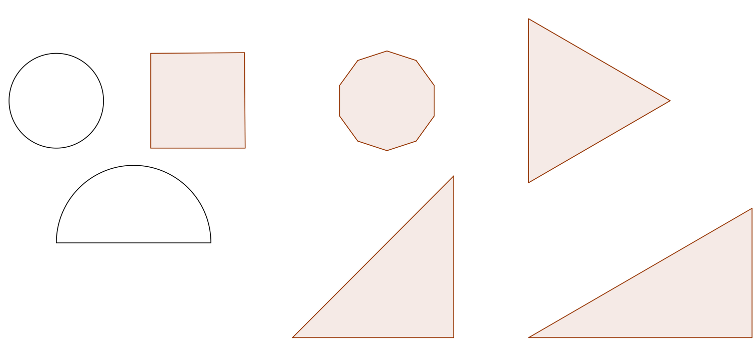

Figure 1: Graphic representation of shapes where area equals perimeter

3 Cases with two parameters

3.1 The isosceles triangle

3.1.1 Functions of the base length

Consider an isosceles triangle, with a as the base, h as the height from the base, and c as the slant height.

Using a and c, solve for h, set area equal to perimeter, solve for c, then replace:

$$\frac{1}{2}a\sqrt{c^2 - a^2/4} = a + 2c \tag{14}$$

$$c = \frac{a(a^2+16)}{2(a^2-16)}, a > 4 \tag{15}$$

$$c = \frac{a}{2} + \frac{8}{a-4} + \frac{8}{a+4}, a > 4 \tag{16}$$

$$h = \frac{4a^2}{a^2-16}, a > 4 \tag{17}$$

$$h = 4 + \frac{8}{a-4} - \frac{8}{a+4}, a > 4 \tag{18}$$

$$K = \frac{2a^3}{a^2-16}, a > 4 \tag{19}$$

$$K = 2a + \frac{16}{a-4} + \frac{16}{a+4}, a > 4 \tag{20}$$

Note the importance of the number 4.

The minimum of c occurs with these values:

$$a = 4\sqrt{2+\sqrt{5}} \approx 8.233$$
$$c = (1+\sqrt{5})\sqrt{2+\sqrt{5}} \approx 6.663$$
$$h = 3+\sqrt{5} \approx 5.236$$
$$K = (6+2\sqrt{5})\sqrt{2+\sqrt{5}} \approx 21.553$$

For this triangle, the ratio of c to a is half of the Golden Ratio. The ratio of c to h is the square root of the Golden Ratio. These ratios are close to those of the Great Pyramid of Giza (Wikipedia: see Figure 2). The right

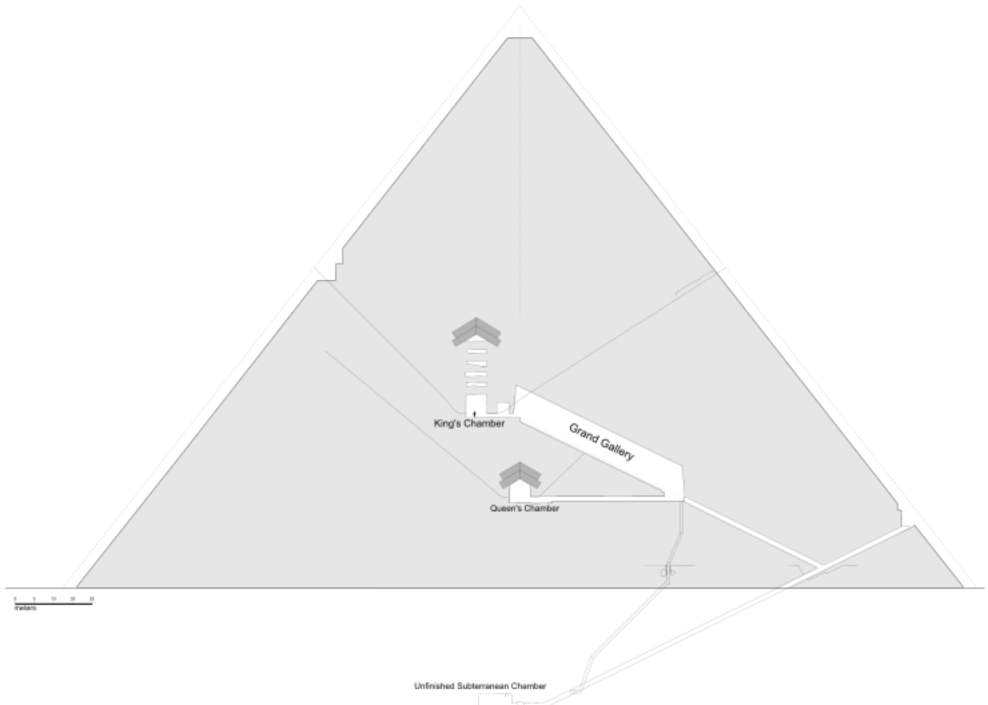

Figure 2: Great Pyramid of Giza (Source:Wikipedia)

triangle formed by the height, the slant height, and half the base is known as the Kepler triangle (Wikipedia).

Since $h(a)$ is decreasing for all a, there is no relative minimum or maximum value for the function, only a horizontal asymptote at 4.

The minimum for K occurs when $a = 4\sqrt{3} \approx 6.928$, which is the equilateral triangle.

The integer solution for K is $a = 12, K = 27, c = 15/2, h = 9/2$. The right triangle formed by the height, the slant height, and half the base is a 3-4-5 triangle.

3.1.2 Functions of the height

$$a = 4\sqrt{\frac{h}{h-4}}, h > 4 \tag{21}$$

$$c = (h-2)\sqrt{\frac{h}{h-4}}, h > 4 \tag{22}$$

$$K = 2h\sqrt{\frac{h}{h-4}}, h > 4 \tag{23}$$

Note the importance of the number 4.

3.2 The right triangle

3.2.1 Functions of leg length

Using the traditional notation, we set the area equal to the perimeter, solve for c, square both sides, combine with the Pythagorean Theorem, and solve for a in terms of b:

$$\frac{1}{2}ab = a + b + c \tag{24}$$

$$a^2 + b^2 = \frac{1}{4}a^2b^2 - a^2b - ab^2 + a^2 + 2ab + b^2 \tag{25}$$

$$a = \frac{4(b-2)}{b-4}, b > 4 \tag{26}$$

$$a = 4 + \frac{8}{b-4}, b > 4 \tag{27}$$

$$(a-4)(b-4) = 8, a, b > 4 \tag{28}$$

$$c = b + \frac{8}{b-4}, b > 4 \tag{29}$$

$$K = 2b + 4 + \frac{16}{b-4}, b > 4 \tag{30}$$

Note the importance of the number 4.

Equation 26 is an example of an involution, a function which is its own inverse. In fact, any rational function of first-degree functions, $f(x) = \frac{ax+b}{cx+d}$, will be an involution when $d = -a$. This function in our context demonstrates the interchangeability of a and b as two legs of the triangle.

Equation 28 allows easy identification of integer solutions, which are $(a, b, c; K) = (5, 12, 13; 30), (6, 8, 10; 24), (8, 6, 10; 24), (12, 5, 13; 30)$. There are also integer solutions for K:

$$(a, b, c; K) = (20, 9/2, 41/2; 45), (9/2, 20, 41/2; 45)$$

.

Note that $a + b = c + 4; K = 2c + 4 = 2(a + b) - 4$.

The minimum values of c and K occur at $a, b = 4 + 2\sqrt{2} \approx 6.828$, which is the isosceles right triangle. Those values are $c = 4(1 + \sqrt{2}) \approx 9.657$ and $K = 4(3 + 2\sqrt{2}) \approx 23.314$.

3.2.2 Functions of the hypotenuse

$$a, b = \frac{c + 4 \pm \sqrt{c^2 - 8c - 16}}{2}, c \geq 4(1 + \sqrt{2}) \tag{31}$$

$$K = 2c + 4, c \geq 4(1 + \sqrt{2}) \tag{32}$$

The values of a and b correspond to the plus and minus values of Equation 31. The function with addition is increasing, and the function with subtraction is decreasing, so the extreme values correspond to the isosceles right triangle.

3.2.3 Functions of the area and perimeter

$$a, b = \frac{K + 4 \pm \sqrt{K^2 - 24K + 16}}{4}, K \geq 4(3 + 2\sqrt{2}) \tag{33}$$

$$c = \frac{K - 4}{2}, K \geq 4(3 + 2\sqrt{2}) \tag{34}$$

The values of a and b correspond to the plus and minus values of Equation 33.

3.3 The rectangle

3.3.1 Functions of a side

$$LW = 2L + 2W \tag{35}$$

$$L = \frac{2W}{W-2}, W > 2 \tag{36}$$

$$L = 2 + \frac{4}{W-2}, W > 2 \tag{37}$$

$$(L-2)(W-2) = 4, L, W > 2 \tag{38}$$

$$K = \frac{2W^2}{W-2}, W > 2 \tag{39}$$

$$K = 2W + 4 + \frac{8}{W-2}, W > 2 \tag{40}$$

In these equations, the number 2 is very important, but the number 4 is also involved.

Equation 36 is an involution, similar to Equation 26. Equation 38 leads to integer solutions $(L, W; K) = (3, 6; 18), (4, 4; 16), (6, 3; 18)$. Equation 40 leads to the integer solutions $(L, W; K) = (5/2, 10; 25), (10, 5/2; 25)$.

The minimum value of K is 16, at $L, W = 4$, which is the square.

3.3.2 Function of the area and perimeter

$$L, W = \frac{K \pm \sqrt{K}\sqrt{K-16}}{4}, K > 16 \tag{41}$$

3.4 The sector of a circle

3.4.1 Functions of the central angle

Using r for radius, and t for the central angle:

$$\frac{1}{2}tr^2 = 2r + tr, 0 < t < 2\pi \tag{42}$$

$$r = \frac{2(t+2)}{t}, 0 < t < 2\pi \tag{43}$$

$$r = 2 + \frac{4}{t}, 0 < t < 2\pi \tag{44}$$

$$(r-2)(t) = 4, 0 < t < 2\pi, r > \frac{2(\pi+1)}{\pi} \tag{45}$$

$$K = \frac{2(t+2)^2}{t}, 0 < t < 2\pi \tag{46}$$

$$K = 2t + 8 + \frac{8}{t}, 0 < t < 2\pi \tag{47}$$

Note the importance of the numbers 2 and 4.

Integer solutions are: $(t, r; K) = (1, 6; 18), (2, 4; 16), (4, 3; 18)$. An integer solution for K is $(t, r; K) = (1/2, 10, 25)$.

The minimum for $K(K = 16)$ occurs when $t = 2, r = 4$.

Note that $2r + 2t + 4 = K$.

See below for connection with the rectangle, which has similar functional behavior.

3.4.2 Functions of the radius

$$t = \frac{4}{r-2}, r > \frac{2(\pi+1)}{\pi} \tag{48}$$

$$K = \frac{2r^2}{r-2}, r > \frac{2(\pi+1)}{\pi} \tag{49}$$

$$K = 2r + 4 + \frac{8}{r-2}, r > \frac{2(\pi+1)}{\pi} \tag{50}$$

3.4.3 Functions of the area and perimeter

$$t = \frac{K - 8 \pm \sqrt{K}\sqrt{K-16}}{4}, r = \frac{K \mp \sqrt{K}\sqrt{K-16}}{4}, 16 \leq K < \frac{4(\pi+1)^2}{\pi}; \tag{51}$$

$$t = \frac{K - 8 - \sqrt{K}\sqrt{K-16}}{4}, r = \frac{K - 8 + \sqrt{K}\sqrt{K-16}}{4}, K \geq \frac{4(\pi+1)^2}{\pi} \tag{52}$$

Note that the addition choice on the variable t is limited, since $t < 2\pi$.

3.4.4 Connection to the rectangle

If the variable t is replaced by $W-2$, r is replaced by L, and the restrictions on the variables are modified as needed, then the equations for the rectangle are obtained.

3.5 The rhombus

3.5.1 Functions of the diagonal

Consider a and b as the diagonals of the rhombus, with c as the side length.

$$\frac{1}{2}ab = 4c = 2\sqrt{a^2+b^2} \tag{53}$$

$$a = \frac{4b}{\sqrt{b^2-16}}, b > 4 \tag{54}$$

$$(a^2-16)(b^2-16) = 256, a, b > 4 \tag{55}$$

$$c = \frac{b^2}{2\sqrt{b^2-16}}, b > 4 \tag{56}$$

$$K = \frac{2b^2}{\sqrt{b^2-16}}, b > 4 \tag{57}$$

Note the importance of the number 4.

Equation 54 is another involution.

The minimum for c and K occur when $a, b = 4\sqrt{2}$, which is the square: $c = 4, K = 16$.

3.5.2 Functions of other parameters

$$a, b = \sqrt{2}\sqrt{c^2 \pm c\sqrt{c^2-16}}, c \geq 4 \tag{58}$$

$$K = 4c, c \geq 4 \tag{59}$$

$$a, b = \frac{\sqrt{2}\sqrt{K^2 \pm K\sqrt{K^2-256}}}{4}, K \geq 16 \tag{60}$$

$$c = K/4, K \geq 16 \tag{61}$$

One interesting note about the rhombus is the measure of the height, which is the perpendicular distance between two parallel sides. Since the area $K = ch = 4c$, then for any rhombus with area equal perimeter, $h = 4$.

4 Conclusion

The exploration of equivalence between area and perimeter numbers provides experience in algebra, functions, and number theory. It also demonstrates a dependence on the number 4, which seems to have a connection to many of the shapes.

5 Further topics

- The effect of rounding decimal numbers
- Compound shapes of various types, including the L-shape, and other combinations (attached, subtracted, inscribed, circumscribed)
- Shapes that require more than two parameters, including the triangle, the parallelogram, and the trapezoid.