100 Integrals

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1 Basic Integrals to Remember

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$

$$\int \sin(x)\, dx = -\cos(x) + C$$

$$\int \cos(x)\, dx = \sin(x) + C$$

$$\int \tan(x)\, dx = -\ln|\cos(x)| + C \text{ or } \ln|\sec(x)| + C$$

$$\int \csc(x)\, dx = -\ln|\csc(x) + \cot(x)| + C$$

$$\int \sec(x)\, dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \cot(x)\, dx = \ln|\sin(x)| + C \text{ or } -\ln|\csc(x)| + C$$

$$\int a^x dx = \frac{1}{\ln(a)}a^x + C$$

$$\int e^x dx = e^x + C$$

$$\int \log_a(x)\, dx = x\log_a(x) - \frac{1}{\ln(a)}x + C$$

$$\int \ln(x)\, dx = x\ln(x) - x + C$$

2 Advanced Integrals to Remember

$$\int \sec^3(x)dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C$$

$$\int \arcsin(x)dx = x\arcsin(x) + \sqrt{1-x^2} + C$$

$$\int \arccos(x)dx = x\arccos(x) - \sqrt{1-x^2} + C$$

$$\int \arctan(x)dx = x\arctan(x) - \frac{1}{2}\ln(x^2+1) + C$$

$$\int \sqrt{x^2+a^2}dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|\sqrt{x^2+a^2}+x\right| + C$$

$$\int \sqrt{x^2-a^2}dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|\sqrt{x^2-a^2}+x\right| + C$$

$$\int \sqrt{a^2-x^2}dx = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2-x^2} + C$$

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left|x+\sqrt{x^2+a^2}\right| + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \sin^n(x)dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx$$

$$\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx$$

3 The 100 Integrals

Problem 1

$$\int \frac{1}{\sqrt{x}\,(1+x)}dx$$

Substituting

$$u = \sqrt{x}$$

$$du = \frac{1}{2\sqrt{x}}dx$$

yields:

$$= 2\int \frac{1}{1+u^2}du$$

This integral is standard:

$$= 2\arctan(u) + C$$

Undoing the substitution(s):

$$= 2\arctan\left(\sqrt{x}\right) + C$$

Problem 2

$$\int \frac{\sec^2(x)}{1+\tan(x)}dx$$

Substituting

$$u = 1 + \tan(x)$$

$$du = \sec^2(x)dx$$

yields:

$$= \int \frac{1}{u}du$$

This integral is standard:

$$= \ln|u| + C$$

Undoing the substitution(s):

$$= \ln|1+\tan(x)| + C$$

Problem 3

$$\int \sin(x) \sec(x) \, dx$$

Rewriting the integral:

$$= \int \frac{\sin(x)}{\cos(x)} dx$$

Substituting

$$u = \cos(x)$$

$$du = -\sin(x) dx$$

yields:

$$= -\int \frac{1}{u} du$$

This integral is standard:

$$= -\int \ln|u| + C$$

Undoing the substitution(s):

$$= -\ln|\cos(x)| + C$$

Problem 4

$$\int \frac{\csc(x) \cot(x)}{1 + \csc^2(x)} dx$$

Substituting

$$u = \csc(x)$$

$$du = -\csc(x) \cot(x) dx$$

yields:

$$= -\int \frac{1}{1 + u^2} du$$

This integral is standard:

$$= -\arctan(u) + C$$

Undoing the substitution(s):

$$= -\arctan(\csc(x)) + C$$

Problem 5

$$\int \frac{\tan(x)}{\cos^2(x)} dx$$

Rewriting the integral:

$$= \int \tan(x) \sec^2(x) dx$$

Substituting

$$u = \tan(x)$$

$$du = \sec^2(x) dx$$

Yields:

$$= \int u du$$

This integral is standard:

$$= \frac{1}{2} u^2 + C$$

Undoing the substitution(s):

$$= \frac{1}{2} \tan^2(x) + C$$

Problem 6

$$\int \csc^4(x) \, dx$$

Rewriting the integral:

$$= \int \csc^2(x) \cdot \csc^2(x) dx$$

$$= \int \csc^2(x)(\cot^2(x) + 1) dx$$

Substituting

$$u = \cot(x)$$

$$du = -\csc^2(x) dx$$

Yields:

$$= -\int (u^2 + 1) du$$

$$= -\int u^2 du - \int du$$

Both integrals are standard:

$$= -\frac{1}{3}u^3 - u + C$$

Undoing the substitution(s):

$$= -\frac{1}{3}\cot^3(x) - \cot(x) + C$$

Problem 7

$$\int x \tan^2(x)\, dx$$

Rewriting the integral:

$$= \int x(\sec^2(x) - 1)dx$$

$$= \int x \sec^2(x)dx - \int xdx$$

Using integration by parts where

$$u = x \qquad v = \tan(x)$$

$$du = dx \qquad dv = \sec^2(x)dx$$

yields:

$$= x\tan(x) - \int \tan(x)dx - \int xdx$$

Both integrals are standard:

$$= x\tan(x) + \ln|\cos(x)| - \frac{1}{2}x^2 + C$$

Problem 8

$$\int x^2 \cos^2(x)\, dx$$

Rewriting the integral:

$$= \int x^2 \left(\frac{1 + \cos(2x)}{2}\right) dx$$

$$= \frac{1}{2}\int x^2(1 + \cos(2x))dx$$

$$= \frac{1}{2}\int x^2 dx + \frac{1}{2}\int x^2\cos(2x)dx$$

Using integration by parts where

Sign	D	I
$+$	x^2	$\cos(2x)$
$-$	$2x$	$\frac{1}{2}\sin(2x)$
$+$	2	$-\frac{1}{4}\cos(2x)$
$-$	0	$-\frac{1}{8}\sin(2x)$

yields:

$$= \frac{1}{6}x^3 + \frac{1}{2}\left(\frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x)\right) + C$$

$$= \frac{1}{6}x^3 + \frac{1}{4}x^2\sin(2x) + \frac{1}{4}x\cos(2x) - \frac{1}{8}\sin(2x) + C$$

Problem 9

$$\int x^5\sqrt{2-x^3}dx$$

Rewriting the integral:

$$= \int x^2 \cdot x^3\sqrt{2-x^3}dx$$

Substituting

$$u = 2 - x^3$$

$$2 - u = x^3$$

$$-du = 3x^2dx$$

yields:

$$= -\frac{1}{3}\int(2-u)\sqrt{u}du$$

$$= -\frac{1}{3}\int\left(2\sqrt{u} - u^{3/2}\right)du$$

$$= -\frac{2}{3}\int\sqrt{u}du + \frac{1}{3}\int u^{3/2}du$$

Both integrals are standard:

$$= -\frac{4}{9}u^{3/2} + \frac{2}{15}u^{5/2} + C$$

Undoing the substitution(s):

$$= -\frac{4}{9}(2-x^3)^{3/2} + \frac{2}{15}(2-x^3)^{5/2} + C$$

Problem 10

$$\int \frac{1}{\sqrt{x^2+4}}dx$$

This integral is standard:

$$= \ln\left|x + \sqrt{x^2+4}\right| + C$$

Problem 11

$$\int \frac{x^2}{\sqrt{25+x^2}}dx$$

Rewriting the integral:

$$= \int \frac{x^2}{\sqrt{25\left(1+\left(\frac{x}{5}\right)^2\right)}}dx$$

$$= \frac{1}{5}\int \frac{x^2}{\sqrt{1+(\frac{x}{5})^2}}dx$$

Substituting

$$\tan\theta = \frac{x}{5}$$

$$5\tan\theta = x$$

$$5\sec^2\theta d\theta = dx$$

yields:

$$= \int \frac{(5\tan\theta)^2}{\sqrt{1+\tan^2\theta}}\sec^2\theta d\theta$$

$$= 25\int \frac{\tan^2\theta\sec^2\theta}{\sqrt{\sec^2\theta}}d\theta$$

$$= 25\int (\sec^2\theta - 1)\sec\theta d\theta$$

$$= 25\int (\sec^3\theta - \sec\theta)d\theta$$

$$= 25\int \sec^3\theta d\theta - 25\int \sec\theta d\theta$$

Both integrals are standard:

$$= 25\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right) - 25\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{25}{2}\sec\theta\tan\theta + \frac{25}{2}\ln|\sec\theta + \tan\theta| - 25\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{25}{2}\sec\theta\tan\theta - \frac{25}{2}\ln|\sec\theta + \tan\theta| + C$$

Undoing the substitution(s):

$$= \frac{25}{2}\cdot\frac{\sqrt{x^2+25}}{5}\cdot\frac{x}{5} - \frac{25}{2}\ln\left|\frac{\sqrt{x^2+25}}{5} + \frac{x}{5}\right| + C$$

$$= \frac{1}{2}x\sqrt{x^2+25} - \frac{25}{2}\ln\left|\sqrt{x^2+25} + x\right| + C$$

Problem 12

$$\int \cos(x)\sqrt{4 - \sin^2(x)}dx$$

Substituting

$$u = \sin(x)$$

$$du = \cos(x)dx$$

yields:

$$= \int \sqrt{4-u^2}du$$

This integral is standard:

$$= 2\arcsin\left(\frac{u}{2}\right) + \frac{1}{2}u\sqrt{4-u^2} + C$$

Undoing the substitution(s):

$$= 2\arcsin\left(\frac{\sin(x)}{2}\right) + \frac{1}{2}\sin(x)\sqrt{4-\sin^2(x)} + C$$

Problem 13

$$\int \frac{1}{x^2 - x + 1}dx$$

Rewriting the integral:

$$= \int \frac{1}{x^2 - x + \frac{1}{4} + \frac{3}{4}} dx$$

$$= 4 \int \frac{1}{4x^2 - 4x + 1 + 3} dx$$

$$= 4 \int \frac{1}{(2x-1)^2 + 3}$$

Substituting

$$u = 2x - 1$$

$$du = 2dx$$

yields:

$$= 2 \int \frac{1}{u^2 + 3} du$$

This integral is standard:

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right) + C$$

Undoing the substitution(s):

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

Problem 14

$$\int \sqrt{x^2 + x + 1} dx$$

Rewriting the integral:

$$= \int \sqrt{x^2 + x + \frac{1}{4} + \frac{3}{4}} dx$$

$$= \frac{1}{2} \int \sqrt{4x^2 + 4x + 1 + 3} dx$$

$$= \frac{1}{2} \int \sqrt{(2x+1)^2 + 3} dx$$

Substituting:

$$u = 2x + 1$$

$$du = 2dx$$

yields:

$$= \frac{1}{4} \int \sqrt{u^2 + 3} du$$

This integral is standard:

$$= \frac{1}{4} \left(\frac{1}{2} u \sqrt{u^2 + 3} + \frac{3}{2} \ln \left| \sqrt{u^2 + 3} + u \right| \right) + C$$

$$= \frac{1}{8} u \sqrt{u^2 + 3} + \frac{3}{8} \ln \left| \sqrt{u^2 + 3} + u \right| + C$$

Undoing the substitution(s):

$$= \frac{2x + 1}{8} \sqrt{4x^2 + 4x + 4} + \frac{3}{8} \ln \left| \sqrt{4x^2 + 4x + 4} + 2x + 1 \right| + C$$

Problem 15

$$\int \frac{5x + 31}{3x^2 - 4x + 11} dx$$

Rewriting the integral:

$$= \int \frac{5x - \frac{10}{3} + \frac{103}{3}}{3x^2 - 4x + 11} dx$$

$$= \frac{5}{6} \int \frac{6x - 4}{3x^2 - 4x + 11} dx + \frac{103}{9} \int \frac{1}{x^2 - \frac{4}{3}x + \frac{11}{3}} dx$$

$$= \frac{5}{6} \int \frac{6x - 4}{3x^2 - 4x + 11} dx + \frac{103}{9} \int \frac{1}{x^2 - \frac{4}{3}x + \frac{4}{9} + \frac{29}{9}} dx$$

$$= \frac{5}{6} \int \frac{6x - 4}{3x^2 - 4x + 11} dx + 103 \int \frac{1}{9x^2 - 12x + 4 + 29} dx$$

$$= \frac{5}{6} \int \frac{6x - 4}{3x^2 - 4x + 11} dx + 103 \int \frac{1}{(3x - 2)^2 + 29}$$

Substituting:

$$u = 3x^2 - 4x + 11 \qquad v = 3x - 2$$

$$du = (6x - 4) dx \qquad dv = 3dx$$

yields:

$$= \frac{5}{6} \int \frac{1}{u} + \frac{103}{3} \int \frac{1}{v^2 + 29} dv$$

Both integrals are standard:

$$= \frac{5}{6} \ln |u| + \frac{103}{3\sqrt{29}} \arctan \left(\frac{v}{\sqrt{29}} \right) + C$$

Undoing the substitution(s):

$$= \frac{5}{6} \ln \left|3x^2 - 4x + 11\right| + \frac{103}{3\sqrt{29}} \arctan\left(\frac{3x-2}{\sqrt{29}}\right) + C$$

Problem 16

$$\int \frac{x^4+1}{x^2+2} dx$$

Rewriting the integral:

$$= \int \frac{x^4 + 2x^2 - 2x^2 - 4 + 5}{x^2+2} dx$$

$$= \int \frac{x^2(x^2+2) - 2(x^2+2) + 5}{x^2+2} dx$$

$$= \int \left(x^2 - 2 + \frac{5}{x^2+2}\right) dx$$

$$= \int x^2 dx - 2\int dx + 5\int \frac{1}{x^2+2} dx$$

All three integrals are standard:

$$= \frac{1}{3}x^3 - 2x + \frac{5}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) + C$$

Problem 17

$$\int \frac{1}{5 + 4\cos(x)} dx$$

Substituting

$$u = \tan\left(\frac{x}{2}\right)$$

$$du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx$$

$$du = \frac{1+u^2}{2} dx$$

$$\frac{2}{1+u^2} du = dx$$

yields:

$$= 2\int \frac{1}{5+4\left(\frac{1-u^2}{1+u^2}\right)} \cdot \frac{1}{1+u^2} du$$

$$= 2\int \frac{1}{5(1+u^2)+4(1-u^2)} du$$

$$= 2\int \frac{1}{5+5u^2+4-4u^2} du$$

$$= 2\int \frac{1}{9+u^2} du$$

This integral is standard:

$$= \frac{2}{3}\arctan\left(\frac{u}{3}\right) + C$$

Undoing the substitution(s):

$$= \frac{2}{3}\arctan\left(\frac{\tan(\frac{x}{2})}{3}\right) + C$$

Problem 18

$$\int \frac{\sqrt{x}}{1+x} dx$$

Substituting

$$\tan\theta = \sqrt{x}$$

$$\tan^2\theta = x$$

$$2\tan\theta\sec^2\theta d\theta = dx$$

yields:

$$= 2\int \frac{\tan\theta}{1+\tan^2\theta} \tan\theta\sec^2\theta d\theta$$

$$= 2\int \frac{\tan^2\theta\sec^2\theta}{\sec^2\theta} d\theta$$

$$= 2\int \left(\sec^2\theta - 1\right) d\theta$$

$$= 2\int \sec^2\theta - 2\int d\theta$$

Both integrals are standard:

$$= 2\tan\theta - 2\theta + C$$

Undoing the substitution(s):

$$= 2\sqrt{x} - 2\arctan\left(\sqrt{x}\right) + C$$

Problem 19

$$\int \frac{\cos(x)}{4 - \sin^2(x)} dx$$

Substituting

$$u = \sin(x)$$

$$du = \cos(x) dx$$

yields:

$$= \int \frac{1}{4 - u^2} du$$

This integral is standard:

$$= \frac{1}{4} \ln \left| \frac{2 + u}{2 - u} \right| + C$$

Undoing the substitution(s):

$$= \frac{1}{4} \ln \left| \frac{2 + \sin(x)}{2 - \sin(x)} \right| + C$$

Problem 20

$$\int \frac{\cos(2x)}{\cos(x)} dx$$

Rewriting the integral:

$$= \int \frac{2\cos^2(x) - 1}{\cos(x)} dx$$

$$= \int (2\cos(x) - \sec(x)) dx$$

$$= 2\int \cos(x) dx - \int \sec(x) dx$$

Both integrals are standard:

$$= 2\sin(x) - \ln|\sec(x) + \tan(x)| + C$$

Problem 21

$$\int \frac{\tan(x)}{\ln(\cos(x))} dx$$

Substituting

$$u = -\ln(\cos(x))$$

$$du = \tan(x) dx$$

yields:

$$= -\int \frac{1}{u} du$$

This integral is standard:

$$= -\ln|u| + C$$

Undoing the substitution(s):

$$= -\ln|-\ln(\cos(x))| + C$$

Problem 22

$$\int \frac{x^7}{\sqrt{1-x^4}} dx$$

Rewriting the integral:

$$= \int \frac{x^3 \cdot x^4}{\sqrt{1-x^4}} dx$$

Substituting:

$$u = 1 - x^4$$

$$1 - u = x^4$$

$$-du = 4x^3 dx$$

Yields:

$$= -\frac{1}{4} \int \frac{1-u}{\sqrt{u}} du$$

$$= -\frac{1}{4} \int \left(u^{-1/2} - \sqrt{u}\right) du$$

$$= -\frac{1}{4} \int u^{-1/2} du + \frac{1}{4} \int \sqrt{u} du$$

Both integrals are standard:

$$= -\frac{1}{2}\sqrt{u} + \frac{1}{6}u^{3/2} + C$$

Undoing the substitution(s):

$$= -\frac{1}{2}\sqrt{1-x^4} + \frac{1}{6}(1-x^4)^{3/2} + C$$

Problem 23

$$\int \ln(1+x)\,dx$$

Substituting

$$u = 1 + x$$

$$du = dx$$

Yields:

$$= \int \ln(u)du$$

This integral is standard:

$$= u\ln(u) - u + C$$

Undoing the substitution(s):

$$= (1+x)\ln(1+x) - (1+x) + C$$

Problem 24

$$\int x \operatorname{arcsec}(x)\,dx$$

Using integration by parts where

$$u = \operatorname{arcsec}(x) \qquad v = \frac{1}{2}x^2$$

$$du = \frac{1}{x\sqrt{x^2-1}}dx \qquad dv = xdx$$

yields:

$$= \frac{1}{2}x^2 \operatorname{arcsec}(x) - \frac{1}{2}\int \frac{x}{\sqrt{x^2-1}}dx$$

Substituting

$$w = x^2 - 1$$

$$dw = 2xdx$$

yields:

$$= -\frac{1}{4}\int \frac{1}{\sqrt{w}}dw$$

This integral is standard:

$$= \frac{1}{2}x^2 \operatorname{arcsec}(x) - \frac{1}{2}\sqrt{w} + C$$

Undoing the substitution(s):

$$= \frac{1}{2}x^2 \operatorname{arcsec}(x) - \frac{1}{2}\sqrt{x^2 - 1} + C$$

Problem 25

$$\int \sqrt{x^2 + 9}dx$$

This integral is standard:

$$= \frac{1}{2}x\sqrt{x^2 + 9} + \frac{9}{2}\ln\left|\sqrt{x^2 + 9} + x\right| + C$$

Problem 26

$$\int \frac{x^2}{\sqrt{4 - x^2}}dx$$

Rewriting the integral:

$$= \int \frac{x^2}{\sqrt{4(1 - \frac{x^2}{4})}}dx$$

$$= \frac{1}{2}\int \frac{x^2}{\sqrt{1 - (\frac{x}{2})^2}}dx$$

Substituting

$$\sin\theta = \frac{x}{2}$$

$$2\sin\theta = x$$

$$2\cos\theta d\theta = dx$$

yields:

$$= \int \frac{(2\sin\theta)^2\cos\theta}{\sqrt{1-\sin^2\theta}}d\theta$$

$$= 4\int \frac{\sin^2\theta\cos\theta}{\sqrt{\cos^2\theta}}d\theta$$

$$= 4\int \sin^2\theta d\theta$$

$$= 4\int \left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= 2\int (1-\cos 2\theta)d\theta$$

$$= 2\int d\theta - 2\int \cos 2\theta d\theta$$

Substituting

$$u = 2\theta$$

$$du = 2d\theta$$

yields:

$$= 2\int d\theta - \int \cos(u)du$$

Both integrals are standard:

$$= 2\theta - \sin(u) + C$$

Undoing the substitution(s):

$$= 2\theta - \sin(2\theta) + C$$

$$= 2\theta - 2\sin\theta\cos\theta + C$$

$$= 2\arcsin\left(\frac{x}{2}\right) - 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C$$

$$= 2\arcsin\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} + C$$

Problem 27

$$\int \sqrt{2x-x^2}dx$$

Rewriting the integral:

$$= \int \sqrt{1 - 1 + 2x - x^2} dx$$

$$= \int \sqrt{1 - (1 - 2x + x^2)} dx$$

$$= \int \sqrt{1 - (1 - x)^2} dx$$

Substituting:

$$u = 1 - x$$

$$du = -dx$$

yields:

$$= - \int \sqrt{1 - u^2} du$$

This integral is standard:

$$= -\frac{1}{2} \arcsin(u) - \frac{1}{2} u \sqrt{1 - u^2} + C$$

Undoing the substitution(s):

$$= -\frac{1}{2} \arcsin(1 - x) - \frac{1}{2} (1 - x) \sqrt{2x - x^2} + C$$

Problem 28

$$\int \frac{4x - 2}{x^3 - x} dx$$

Factoring the integral:

$$= \int \frac{4x - 2}{x(x - 1)(x + 1)} dx$$

Using partial fraction decomposition

$$= \int \left(\frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 1} \right) dx$$

where $A, B, C \in \mathbb{R}$ yields:

$$= \int \left(\frac{2}{x} + \frac{1}{x - 1} + \frac{-3}{x + 1} \right) dx$$

$$= 2 \int \frac{1}{x} dx + \int \frac{1}{x - 1} dx - 3 \int \frac{1}{x + 1} dx$$

Substituting

$$u = x - 1 \qquad\qquad v = x + 1$$

$$du = dx \qquad\qquad dv = dx$$

yields:

$$2\int \frac{1}{x}dx + \int \frac{1}{u}du - 3\int \frac{1}{v}dv$$

All integrals are standard:

$$= 2\ln|x| + \ln|u| - 3\ln|v| + C$$

$$= \ln\left|\frac{x^2u}{v^3}\right| + C$$

Undoing the substitution(s):

$$= \ln\left|\frac{x^2(x-1)}{(x+1)^3}\right| + C$$

Problem 29

$$\int \frac{x^4}{x^2-2}dx$$

Rewriting the integral:

$$= \int \frac{x^4 - 2x^2 + 2x^2 - 4 + 4}{x^2-2}dx$$

$$= \int \left(\frac{x^4-2x^2}{x^2-2} + \frac{2x^2-4}{x^2-2} + \frac{4}{x^2-2}\right)dx$$

$$= \int \frac{x^2(x^2-2)}{x^2-2}dx + \int \frac{2(x^2-2)}{x^2-2}dx + 4\int \frac{1}{x^2-2}dx$$

$$= \int x^2dx + 2\int dx + 4\int \frac{1}{x^2-2}dx$$

All three integrals are standard:

$$= \frac{1}{3}x^3 + 2x + \sqrt{2}\ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + C$$

Problem 30

$$\int \frac{\sec(x)\tan(x)}{\sec(x) + \sec^2(x)}dx$$

Rewriting the integral:

$$= \int \frac{\tan(x)}{1 + \sec(x)} \cdot \frac{\cos(x)}{\cos(x)} dx$$

$$= \int \frac{\sin(x)}{\cos(x) + 1} dx$$

Substituting

$$u = \cos(x) + 1$$

$$du = -\sin(x) dx$$

yields:

$$= -\int \frac{1}{u} du$$

This integral is standard:

$$= -\ln|u| + C$$

Undoing the substitution(s):

$$= -\ln|\cos(x) + 1| + C$$

Problem 31

$$\int \frac{x}{(x^2 + 2x + 2)^2} dx$$

Rewriting the integral:

$$= \int \frac{x}{(x^2 + 2x + 1 + 1)^2} dx$$

$$= \int \frac{x}{((x+1)^2 + 1)^2} dx$$

Substituting

$$\tan\theta = x + 1$$

$$\tan\theta - 1 = x$$

$$\sec^2\theta d\theta = dx$$

yields:

$$= \int \frac{\tan\theta - 1}{(\tan^2\theta + 1)^2}\sec^2\theta d\theta$$

$$= \int \frac{\tan\theta - 1}{\sec^2\theta} \cdot \frac{\cos^2\theta}{\cos^2\theta} d\theta$$

$$= \int \left(\sin\theta\cos\theta - \cos^2\theta\right)$$

$$= \int \left(\sin\theta\cos\theta - \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \int \sin\theta\cos\theta d\theta - \frac{1}{2}\int d\theta - \frac{1}{2}\int \cos 2\theta d\theta$$

Substituting:

$$u = \sin\theta \qquad v = 2\theta$$

$$du = \cos\theta d\theta \qquad dv = 2d\theta$$

yields:

$$= \int u du - \frac{1}{2}\int d\theta - \frac{1}{4}\int \cos(u) du$$

All three integrals are standard:

$$= \frac{1}{2}u^2 - \frac{1}{2}\theta - \frac{1}{4}\sin(u) + C$$

Undoing the substitution(s):

$$= \frac{1}{2}\sin^2\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

$$= \frac{1}{2}\sin^2\theta - \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right)^2 - \frac{1}{2}\arctan(x+1) - \frac{1}{2} \cdot \frac{x+1}{\sqrt{x^2+2x+2}} \cdot \frac{1}{\sqrt{x^2+2x+2}} + C$$

$$= \frac{x^2+2x+1}{2(x^2+2x+2)} - \frac{x+1}{2(x^2+2x+2)} - \frac{1}{2}\arctan(x+1) + C$$

$$= \frac{x^2+x}{2x^2+4x+4} - \frac{1}{2}\arctan(x+1) + C$$

Problem 32

$$\int \frac{x^{1/3}}{x^{1/2} + x^{1/4}} dx$$

Substituting

$$u^3 = \sqrt[4]{x}$$

$$u^4 = \sqrt[3]{x}$$

$$u^6 = \sqrt{x}$$

$$u^{12} = x$$

$$12u^{11} du = dx$$

yields:

$$= \int \frac{u^4}{u^6 + u^3} 12u^{11} du$$

$$= 12 \int \frac{u^{12}}{u^3 + 1} du$$

Performing long division on the integral yields:

$$= 12 \int \left(u^9 - u^6 + u^3 - 1 + \frac{1}{u^3 + 1} \right) du$$

$$= 12 \int u^9 du - 12 \int u^6 du + 12 \int u^3 du - 12 \int du + 12 \int \frac{1}{u^3 + 1} du$$

Factoring the last integral:

$$= 12 \int \frac{1}{(u+1)(u^2 - u + 1)} du$$

Using partial fraction decomposition

$$= 12 \int \left(\frac{A}{u+1} + \frac{Bu + C}{u^2 - u + 1} \right) du$$

where $A, B, C \in \mathbb{R}$ yields:

$$= 12 \int \left(\frac{\frac{1}{3}}{u+1} + \frac{-\frac{1}{3}u + \frac{2}{3}}{u^2 - u + 1} \right) du$$

$$= 4 \int \frac{1}{u+1} du - 4 \int \frac{u - 2}{u^2 - u + 1} du$$

Rewriting the second integral:

$$= 4\int \frac{1}{u+1}du - 4\int \frac{u-\frac{1}{2}-\frac{3}{2}}{u^2-u+1}du$$

$$= 4\int \frac{1}{u+1}du - 4\int \frac{u-\frac{1}{2}}{u^2-u+1}du + 4\int \frac{\frac{3}{2}}{u^2-u+1}du$$

$$= 4\int \frac{1}{u+1}du - 2\int \frac{2u-1}{u^2-u+1}du + 6\int \frac{1}{u^2-u+\frac{1}{4}+\frac{3}{4}}du$$

$$= 4\int \frac{1}{u+1}du - 2\int \frac{2u-1}{u^2-u+1}du + 24\int \frac{1}{4u^2-4u+1+3}du$$

$$= 4\int \frac{1}{u+1}du - 2\int \frac{2u-1}{u^2-u+1}du + 24\int \frac{1}{(2u-1)^2+3}du$$

Substituting

$$v = u+1 \qquad w = u^2-u+1 \qquad y = 2u-1$$

$$dv = du \qquad dw = (2u-1)du \qquad dy = 2du$$

yields:

$$= 4\int \frac{1}{v}dv - 2\int \frac{1}{w}dw + 12\int \frac{1}{y^2+3}dy$$

All three integrals are standard:

$$= 4\ln|v| - 2\ln|w| + 4\sqrt{3}\arctan\left(\frac{y}{\sqrt{3}}\right) + C$$

$$= \ln\left|v^4\right| - \ln\left|w^2\right| + 4\sqrt{3}\arctan\left(\frac{y}{\sqrt{3}}\right) + C$$

$$= \ln\left|\frac{v^4}{w^2}\right| + 4\sqrt{3}\arctan\left(\frac{y}{\sqrt{3}}\right) + C$$

The previous four integrals are also standard:

$$= \frac{6}{5}u^{10} - \frac{12}{7}u^7 + 3u^4 - 12u + \ln\left|\frac{v^4}{w^2}\right| + 4\sqrt{3}\arctan\left(\frac{y}{\sqrt{3}}\right) + C$$

Undoing the substitution(s):

$$= \frac{6}{5}u^{10} - \frac{12}{7}u^7 + 3u^4 - 12u + \ln\left|\frac{(u+1)^4}{(u^2-u+1)^2}\right| + 4\sqrt{3}\arctan\left(\frac{2u-1}{\sqrt{3}}\right) + C$$

$$= \frac{6}{5}x^{5/6} - \frac{12}{7}x^{7/12} + 3x^{1/3} - 12x^{1/12} + \cdots$$

$$+ \ln\left|\frac{(x^{1/12}+1)^4}{(x^{1/6}-x^{1/12}+1)^2}\right| + 4\sqrt{3}\arctan\left(\frac{2x^{1/12}-1}{\sqrt{3}}\right) + C$$

Problem 33

$$\int \frac{1}{1+\cos(2x)}dx$$

Rewriting the integral:

$$= \int \frac{1}{1+2\cos^2(x)-1}dx$$

$$= \frac{1}{2}\int \sec^2(x)dx$$

This integral is standard:

$$= \frac{1}{2}\tan(x) + C$$

Problem 34

$$\int \frac{\sec(x)}{\tan(x)}dx$$

Rewriting the integral:

$$= \int \frac{\sec(x)}{\tan(x)} \cdot \frac{\cos(x)}{\cos(x)}dx$$

$$= \int \frac{1}{\sin(x)}dx$$

$$= \int \csc(x)dx$$

This integral is standard:

$$= -\ln|\cot(x) + \csc(x)| + C$$

Problem 35

$$\int \sec^3(x)\tan^3(x)\,dx$$

Rewriting the integral:

$$= \int \sec^2(x)\tan^2(x)\sec(x)\tan(x)dx$$

$$= \int \sec^2(x)(\sec^2(x) - 1)\sec(x)\tan(x)$$

$$= \int (\sec^4(x) - \sec^2(x))\sec(x)\tan(x)$$

Substituting

$$u = \sec(x)$$

$$du = \sec(x)\tan(x)dx$$

yields:

$$= \int \left(u^4 - u^2\right) du$$

$$= \int u^4 du - \int u^2 du$$

Both integrals are standard:

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

Undoing the substitution(s):

$$= \frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C$$

Problem 36

$$\int x^2 \arctan(x)\,dx$$

Using integration by parts where

$$u = \arctan(x) \qquad v = \frac{1}{3}x^3$$

$$du = \frac{1}{x^2+1}dx \qquad dv = x^2 dx$$

yields:

$$= \frac{1}{3}x^3\arctan(x) - \frac{1}{3}\int \frac{x^3}{x^2+1}dx$$

Problem 37

$$\int x \ln^3(x)\, dx$$

Substituting

$$u = \ln(x)$$

$$e^u = x$$

$$e^u du = dx$$

yields:

$$= \int u^3 e^{2u} du$$

Using integration by parts where

Sign	D	I
$+$	u^3	e^{2u}
$-$	$3u^2$	$\frac{1}{2}e^{2u}$
$+$	$6u$	$\frac{1}{4}e^{2u}$
$-$	6	$\frac{1}{8}e^{2u}$
$+$	0	$\frac{1}{16}e^{2u}$

yields:

$$= \frac{1}{2}u^3 e^{2u} - \frac{3}{4}u^2 e^{2u} + \frac{3}{4}ue^{2u} - \frac{3}{8}e^{2u} + C$$

Undoing the substitution(s):

$$= \frac{1}{2}x^2 \ln^3(x) - \frac{3}{4}x^2 \ln^2(x) + \frac{3}{4}x^2 \ln(x) - \frac{3}{8}x^2 + C$$

Problem 38

$$\int \frac{1}{x\sqrt{1+x^2}} dx$$

Substituting

$$\tan\theta = x$$

$$\sec^2\theta d\theta dx$$

yields:

$$= \int \frac{\sec^2\theta}{\tan\theta\sqrt{1+\tan^2\theta}} d\theta$$

$$= \int \frac{\sec^2\theta}{\tan\theta\sqrt{\sec^2\theta}} d\theta$$

$$= \int \frac{\sec\theta}{\tan\theta} d\theta$$

$$= \int \frac{\sec\theta}{\tan\theta} \cdot \frac{\cos\theta}{\cos\theta} d\theta$$

$$= \int \frac{1}{\sin\theta} d\theta$$

$$= \int \csc\theta d\theta$$

This integral is standard:

$$= -\ln|\cot\theta + \csc\theta| + C$$

Undoing the substitution(s):

$$= -\ln\left|\frac{1}{x} + \frac{\sqrt{x^2+1}}{x}\right| + C$$

$$= -\ln\left|\frac{1+\sqrt{x^2+1}}{x}\right| + C$$

$$= \ln\left|\frac{x}{1+\sqrt{x^2+1}}\right| + C$$

Problem 39

$$\int e^x\sqrt{1+e^{2x}}dx$$

Substituting

$$\tan\theta = e^x$$

$$\sec^2\theta d\theta = e^x dx$$

yields:

$$= \int \sec^2\theta\sqrt{1+\tan^2\theta}d\theta$$

$$= \int \sec^2\theta\sqrt{\sec^2\theta}d\theta$$

$$= \int \sec^3\theta d\theta$$

This integral is standard:

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

Undoing the substitution(s):

$$= \frac{1}{2}e^x\sqrt{1+e^{2x}} + \frac{1}{2}\ln\left|\sqrt{1+e^{2x}}+e^x\right| + C$$

Problem 40

$$\int \frac{x}{4x-x^2}dx$$

Rewriting the integral:

$$= \int \frac{1}{4-x}dx$$

Substituting

$$u = 4-x$$

$$du = -dx$$

yields:

$$= -\int \frac{1}{u}du$$

This integral is standard:

$$= -\ln|u| + C$$

Undoing the substitution(s):

$$= -\ln|4-x| + C$$

Problem 41

$$\int \frac{1}{x^3\sqrt{x^2-9}}dx$$

Rewriting the integral:

$$= \int \frac{1}{x^3\sqrt{9\left(\frac{x^2}{9}-1\right)}}dx$$

$$=\frac{1}{3}\int \frac{1}{x^3\sqrt{\left(\frac{x}{3}\right)^2-1}}dx$$

Substituting

$$\sec\theta = \frac{x}{3}$$

$$3\sec\theta = x$$

$$3\sec\theta\tan\theta d\theta = dx$$

yields:

$$= \frac{1}{3}\int \frac{3\sec\theta\tan\theta}{(3\sec\theta)^3\sqrt{\sec^2\theta-1}}d\theta$$

$$= \frac{1}{27}\int \frac{\tan\theta}{\sec^2\theta\sqrt{\tan^2\theta}}d\theta$$

$$= \frac{1}{27}\int \cos^2\theta d\theta$$

$$= \frac{1}{54}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{54}\int d\theta + \frac{1}{54}\int \cos 2\theta d\theta$$

Substituting

$$u = 2\theta$$

$$du2d\theta$$

yields:

$$= \frac{1}{54}\int d\theta + \frac{1}{108}\int \cos(u)du$$

Both integrals are standard:

$$= \frac{1}{54}\theta + \frac{1}{108}\sin(u) + C$$

Undoing the substitution(s):

$$= \frac{1}{54}\theta + \frac{1}{108}\sin 2\theta + C$$

$$= \frac{1}{54}\theta + \frac{1}{54}\sin\theta\cos\theta + C$$

$$= \frac{1}{54}\operatorname{arcsec}\left(\frac{x}{3}\right) + \frac{1}{54}\cdot\frac{\sqrt{x^2-9}}{x}\cdot\frac{3}{x} + C$$

$$= \frac{1}{54}\operatorname{arcsec}\left(\frac{x}{3}\right) + \frac{\sqrt{x^2-9}}{18x^2} + C$$

Problem 42

$$\int \frac{x}{(7x+1)^{17}}dx$$

Substituting

$$u = 7x + 1$$

$$\frac{1}{7}u - \frac{1}{7} = x$$

$$\frac{1}{7}du = dx$$

yields:

$$= \frac{1}{7}\int \frac{\frac{1}{7}u - \frac{1}{7}}{u^{17}}du$$

$$= \frac{1}{49}\int \frac{u-1}{u^{17}}du$$

$$= \frac{1}{49}\int \left(u^{-16} - u^{-17}\right)du$$

$$= \frac{1}{49}\int u^{-16}du - \frac{1}{49}\int u^{-17}du$$

Both integrals are standard:

$$= -\frac{1}{735u^{15}} + \frac{1}{784u^{16}} + C$$

Undoing the substitution(s):

$$= -\frac{1}{735(7x+1)^{15}} + \frac{1}{784(7x+1)^{16}} + C$$

Problem 43

$$\int \frac{4x^2 + x + 1}{4x^3 + x} dx$$

Rewriting the integral:

$$= \int \frac{4x^2 + 1 + x}{x(4x^2 + 1)} dx$$

$$= \int \left(\frac{4x^2 + 1}{x(4x^2 + 1)} + \frac{x}{x(4x^2 + 1)} \right) dx$$

$$= \int \left(\frac{1}{x} + \frac{1}{4x^2 + 1} \right) dx$$

$$= \int \frac{1}{x} dx + \int \frac{1}{4x^2 + 1} dx$$

Substituting

$$u = 2x$$

$$du = 2dx$$

yields:

$$= \int \frac{1}{x} dx + \frac{1}{2} \int \frac{1}{u^2 + 1} du$$

Both integrals are standard:

$$= \ln|x| + \frac{1}{2} \arctan(u) + C$$

Undoing the substitution(s):

$$= \ln|x| + \frac{1}{2} \arctan(2x) + C$$

Problem 44

$$\int \frac{4x^3 - x + 1}{x^3 + 1} dx$$

Performing long division on the integral yields:

$$= \int \left(4 - \frac{x + 3}{x^3 + 1} \right) dx$$

$$= 4 \int dx - \int \frac{x + 3}{(x + 1)(x^2 - x + 1)} dx$$

Using partial fraction decomposition

$$= -\int \left(\frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \right)$$

where $A, B, C \in \mathbb{R}$ yields:

$$= -\int \left(\frac{\frac{2}{3}}{x+1} + \frac{-\frac{2}{3}x + \frac{7}{3}}{x^2-x+1} \right) dx$$

$$= -\frac{1}{3} \int \left(\frac{2}{x+1} - \frac{2x-7}{x^2-x+1} \right) dx$$

$$= -\frac{2}{3} \int \frac{1}{x+1} dx + \frac{1}{3} \int \frac{2x-7}{x^2-x+1} dx$$

$$= -\frac{2}{3} \int \frac{1}{x+1} dx + \frac{1}{3} \int \frac{2x-1-6}{x^2-x+1} dx$$

$$= -\frac{2}{3} \int \frac{1}{x+1} dx + \frac{1}{3} \int \left(\frac{2x-1}{x^2-x+1} - \frac{6}{x^2-x+1} \right) dx$$

$$= -\frac{2}{3} \int \frac{1}{x+1} dx + \frac{1}{3} \int \frac{2x-1}{x^2-x+1} dx - 2 \int \frac{1}{x^2-x+\frac{1}{4}+\frac{3}{4}} dx$$

$$= -\frac{2}{3} \int \frac{1}{x+1} dx + \frac{1}{3} \int \frac{2x-1}{x^2-x+1} dx - 8 \int \frac{1}{4x^2-4x+1+3} dx$$

$$= -\frac{2}{3} \int \frac{1}{x+1} dx + \frac{1}{3} \int \frac{2x-1}{x^2-x+1} dx - 8 \int \frac{1}{(2x-1)^2+3} dx$$

Substituting

$$u = x+1 \qquad v = x^2 - x + 1 \qquad w = 2x - 1$$

$$du = dx \qquad dv = (2x-1)dx \qquad dw = 2dx$$

yields:

$$= -\frac{2}{3} \int \frac{1}{u} du + \frac{1}{3} \int \frac{1}{v} dv - 4 \int \frac{1}{w^2+3} dw$$

All three integrals are standard:

$$= -\frac{2}{3} \ln |u| + \frac{1}{3} \ln |v| - \frac{4}{\sqrt{3}} \arctan \left(\frac{w}{\sqrt{3}} \right) + C$$

$$= -\frac{1}{3} \ln \left| u^2 \right| + \frac{1}{3} \ln |v| - \frac{4}{\sqrt{3}} \arctan \left(\frac{w}{\sqrt{3}} \right) + C$$

$$= \frac{1}{3} \ln \left| \frac{v}{u^2} \right| - \frac{4}{\sqrt{3}} \arctan \left(\frac{w}{\sqrt{3}} \right) + C$$

Undoing the substitution(s):

$$= 4x + \frac{1}{3}\ln\left|\frac{x^2 - x + 1}{(x+1)^2}\right| - \frac{4}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

Problem 45

$$\int \tan^2(x)\sec(x)\,dx$$

Rewriting the integral:

$$= \int (\sec^2(x) - 1)\sec(x)dx$$

$$= \int (\sec^3(x) - \sec(x))dx$$

$$= \int \sec^3(x)dx - \int \sec(x)dx$$

Both integrals are standard:

$$= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x) - \ln|\sec(x) + \tan(x)| + C$$

$$= \frac{1}{2}\sec(x)\tan(x) - \frac{1}{2}\ln|\sec(x) - \tan(x)| + C$$

Problem 46

$$\int \frac{x^2 + 2x + 2}{(x+1)^3}dx$$

Rewriting the integral:

$$= \int \frac{x^2 + 2x + 1 + 1}{(x+1)^3}dx$$

$$= \int \left(\frac{x^2 + 2x + 1}{(x+1)^3} + \frac{1}{(x+1)^3}\right)dx$$

$$= \int \left(\frac{(x+1)^2}{(x+1)^3} + \frac{1}{(x+1)^3}\right)dx$$

$$= \int \frac{1}{x+1}dx + \int \frac{1}{(x+1)^3}dx$$

Substituting

$$u = x + 1$$

$$du = dx$$

Yields:

$$= \int \frac{1}{u} du + \int \frac{1}{u^3} du$$

Both integrals are standard:

$$= \ln |u| - \frac{1}{2u^2} + C$$

Undoing the substitution(s):

$$= \ln |x+1| - \frac{1}{2(x+1)^2} + C$$

Problem 47

$$\int \frac{x^4 + 2x + 2}{x^5 + x^4} dx$$

Rewriting the integral:

$$= \int \frac{x^4 + 2(x+1)}{x^5 + x^4} dx$$

$$= \int \left(\frac{x^4}{x^4(x+1)} + \frac{2(x+1)}{x^4(x+1)} \right) dx$$

$$= \int \frac{1}{x+1} dx + 2 \int \frac{1}{x^4} dx$$

Substituting

$$u = x + 1$$

$$du = dx$$

yields:

$$= \int \frac{1}{u} du + 2 \int \frac{1}{x^4} dx$$

Both integrals are standard:

$$= \ln |u| - \frac{2}{3x^3} + C$$

Undoing the substitution(s):

$$= \ln |x+1| - \frac{2}{3x^3} + C$$

Problem 48

$$\int \frac{8x^2 - 4x + 7}{(x^2+1)(4x+1)} dx$$

Rewriting the integral:

$$= \int \frac{8x^2 + 8 - 4x - 1}{(x^2+1)(4x+1)} dx$$

$$= \int \frac{8(x^2+1) - (4x+1)}{(x^2+1)(4x+1)}$$

$$= \int \left(\frac{8(x^2+1)}{(x^2+1)(4x+1)} - \frac{4x+1}{(x^2+1)(4x+1)} \right) dx$$

$$= \int \left(\frac{8}{4x+1} - \frac{1}{x^2+1} \right) dx$$

$$= 8 \int \frac{1}{4x+1} - \int \frac{1}{x^2+1} dx$$

Substituting

$$u = 4x + 1$$

$$du = 4dx$$

yields:

$$= 2 \int \frac{1}{u} du - \int \frac{1}{x^2+1} dx$$

Both integrals are standard:

$$= 2 \ln |u| - \arctan(x) + C$$

Undoing the substitution(s):

$$= 2 \ln |4x+1| - \arctan(x) + C$$

Problem 49

$$\int \frac{3x^5 - x^4 + 2x^3 - 12x^2 - 2x + 1}{(x^3-1)^2} dx$$

Factoring the integral:

$$= \int \frac{3x^5 - x^4 + 2x^3 - 12x^2 - 2x + 1}{((x-1)(x^2+x+1))^2} dx$$

$$= \int \frac{3x^5 - x^4 + 2x^3 - 12x^2 - 2x + 1}{(x-1)^2(x^2+x+1)^2} dx$$

Using partial fraction decomposition

$$= \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{(x^2+x+1)^2} \right) dx$$

where $A, B, C, D, E, F \in \mathbb{R}$ yields:

$$= \int \left(\frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{2x+1}{x^2+x+1} + \frac{4x+2}{(x^2+x+1)^2} \right) dx$$

$$= \int \frac{1}{x-1} dx - \int \frac{1}{(x-1)^2} dx + \int \frac{2x+1}{x^2+x+1} dx + 2 \int \frac{2x+1}{(x^2+x+1)^2} dx$$

Substituting

$$u = x - 1 \qquad v = x^2 + x + 1$$

$$du = dx \qquad dv = (2x+1)dx$$

yields:

$$= \int \frac{1}{u} du - \int \frac{1}{u^2} du + \int \frac{1}{v} dv + 2 \int \frac{1}{v^2} dv$$

All four integrals are standard:

$$= \ln|u| + \frac{1}{u} + \ln|v| - \frac{2}{v} + C$$

$$= \ln|uv| + \frac{v}{uv} - \frac{2u}{uv} + C$$

$$= \ln|uv| + \frac{v-2u}{uv} + C$$

Undoing the substitution(s):

$$= \ln|(x-1)(x^2+x+1)| + \frac{x^2+x+1-2(x-1)}{(x-1)(x^2+x+1)} + C$$

$$= \ln\left|x^3-1\right| + \frac{x^2-x+3}{x^3-1} + C$$

Problem 50

$$\int \frac{x}{x^4+4x^2+8} dx$$

Rewriting the integral:

$$= \int \frac{x}{x^4+4x^2+4+4} dx$$

$$= \int \frac{x}{(x^2+2)^2+4} dx$$

Substituting

$$u = x^2 + 2$$

$$du = 2xdx$$

yields:

$$= \frac{1}{2} \int \frac{1}{u^2 + 4} du$$

This integral is standard:

$$= \frac{1}{4} \arctan\left(\frac{u}{2}\right) + C$$

Undoing the substitution(s):

$$= \frac{1}{4} \arctan\left(\frac{x^2 + 2}{2}\right) + C$$

Problem 51

$$\int \frac{1}{4 + 5\cos(x)} dx$$

Substituting

$$u = \tan\left(\frac{x}{2}\right)$$

$$du = \frac{1}{2} \sec^2\left(\frac{x}{2}\right) dx$$

$$\frac{2}{1 + u^2} du = dx$$

yields:

$$= \int \frac{1}{4 + 5\left(\frac{1-u^2}{1+u^2}\right)} \cdot \frac{2}{1 + u^2} du$$

$$= 2 \int \frac{1}{4(1 + u^2) + 5(1 - u^2)} du$$

$$= 2 \int \frac{1}{4 + 4u^2 + 5 - 5u^2} du$$

$$= 2 \int \frac{1}{9 - u^2} du$$

This integral is standard:

$$= \frac{1}{3} \ln\left|\frac{3 + u}{3 - u}\right| + C$$

Undoing the substitution(s):

$$= \frac{1}{3} \ln \left| \frac{3 + \tan(\frac{x}{2})}{3 - \tan(\frac{x}{2})} \right| + C$$

Problem 52

$$\int \frac{\left(1 + x^{2/3}\right)^{3/2}}{x^{1/3}} dx$$

Substituting

$$u = 1 + x^{2/3}$$

$$du = \frac{2}{3} x^{-1/3} dx$$

yields:

$$= \frac{3}{2} \int u^{3/2} du$$

This integral is standard:

$$= \frac{3}{5} u^{5/2} + C$$

Undoing the substitution(s):

$$= \frac{3}{5}(1 + x^{2/3})^{5/2} + C$$

Problem 53

$$\int \frac{\arcsin^2(x)}{\sqrt{1 - x^2}} dx$$

Substituting

$$u = \arcsin(x)$$

$$du = \frac{1}{\sqrt{1 - x^2}} dx$$

yields:

$$= \int u^2 du$$

This integral is standard:

$$= \frac{1}{3} u^3 + C$$

Undoing the substitution(s):

$$= \frac{1}{3} \arcsin^3(x) + C$$

Problem 54

$$\int \frac{1}{x^{3/2}\left(1+x^{1/3}\right)}dx$$

Substituting

$$u = x^{1/6}$$

$$u^2 = x^{1/3}$$

$$u^6 = x$$

$$u^9 = x^{3/2}$$

$$6u^5du = dx$$

yields:

$$= \int \frac{1}{u^9(1+u^2)}6u^5du$$

$$= 6\int \frac{1}{u^4(1+u^2)}du$$

Using partial fraction decomposition

$$= \int \left(\frac{A}{u} + \frac{B}{u^2} + \frac{C}{u^3} + \frac{D}{u^4} + \frac{Eu+F}{u^2+1}\right) du$$

where $A, B, C, D, E, F \in \mathbb{R}$ yields:

$$= 6\int \left(-\frac{1}{u^2} + \frac{1}{u^4} + \frac{1}{u^2+1}\right) du$$

$$= -6\int \frac{1}{u^2}du + 6\int \frac{1}{u^4}du + 6\int \frac{1}{u^2+!}du$$

All three integrals are standard:

$$= \frac{6}{u} - \frac{2}{u^3} + 6\arctan(u) + C$$

Undoing the substitution(s):

$$= \frac{6}{x^{1/6}} - \frac{2}{\sqrt{x}} + 6\arctan\left(x^{1/6}\right) + C$$

Problem 55

$$\int \tan^3(x)\,dx$$

Rewriting the integral:

$$= \int \tan^2(x)\tan(x)dx$$

$$= \int (\sec^2(x) - 1)\tan(x)dx$$

$$= \int (\sec^2(x)\tan(x) - \tan(x))dx$$

$$= \int \sec^2(x)\tan(x)dx - \int \tan(x)dx$$

Substituting

$$u = \tan(x)$$

$$du = \sec^2(x)dx$$

yields:

$$= \int u du - \int \tan(x)dx$$

Both integrals are standard:

$$= \frac{1}{2}u^2 + \ln|\cos(x)| + C$$

Undoing the substitution(s):

$$= \frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C$$

Problem 56

$$\int \sin^2(x)\cos^4(x)\,dx$$

Rewriting the integral:

$$= \int (1 - \cos^2(x))\cos^4(x)dx$$

$$= \int (\cos^4(x) - \cos^6(x))dx$$

$$= \int \cos^4(x)dx - \int \cos^6(x)dx$$

Using the reduction rule for $\cos^n(x)$, it's known that:

$$\int \cos^4(x)dx = \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\int \cos^2(x)dx$$

$$= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\int \frac{1+\cos(2x)}{2}dx$$

$$= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{8}\int (1+\cos(2x))dx$$

$$= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{8}\int dx + \frac{3}{8}\int \cos(2x)dx$$

$$= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{8}x + \frac{3}{16}\sin(2x) + C$$

$$= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{8}\sin(x)\cos(x) + \frac{3}{8}x + C$$

and

$$\int \cos^6(x)dx = \frac{1}{6}\cos^5(x)\sin(x) + \frac{5}{6}\int \cos^4(x)dx$$

Rewriting the integral:

$$= \int \cos^4(x)dx - \frac{1}{6}\cos^5(x)\sin(x) - \frac{5}{6}\int \cos^4(x)dx$$

$$= -\frac{1}{6}\cos^5(x)\sin(x) + \frac{1}{6}\int \cos^4(x)dx$$

$$= -\frac{1}{6}\cos^5(x)\sin(x) + \frac{1}{6}\left(\frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{8}\sin(x)\cos(x) + \frac{3}{8}x + C\right)$$

$$= -\frac{1}{6}\cos^5(x)\sin(x) + \frac{1}{24}\cos^3(x)\sin(x) + \frac{1}{16}\sin(x)\cos(x) + \frac{1}{16}x + C$$

Problem 57

$$\int \frac{xe^{x^2}}{1+e^{2x^2}}dx$$

Substituting

$$u = e^{x^2}$$

$$du = 2xe^{x^2}$$

yields:

$$= \frac{1}{2}\int \frac{1}{1+u^2}du$$

This integral is standard:

$$= \frac{1}{2}\arctan(u) + C$$

Undoing the substitution(s):

$$= \frac{1}{2}\arctan\left(e^{x^2}\right) + C$$

Problem 58

$$\int \frac{\cos^3(x)}{\sqrt{\sin(x)}}dx$$

Rewriting the integral:

$$= \int \frac{\cos(x)\cos^2(x)}{\sqrt{\sin(x)}}dx$$

$$= \int \frac{\cos(x)(1-\sin^2(x))}{\sqrt{\sin(x)}}dx$$

Substituting

$$u = \sin(x)$$

$$du = \cos(x)dx$$

yields:

$$= \int \frac{1-u^2}{\sqrt{u}}du$$

$$= \int \left(\frac{1}{\sqrt{u}} - u^{3/2}\right)du$$

$$= \int \frac{1}{\sqrt{u}}du - \int u^{3/2}du$$

Both integrals are standard:

$$= 2\sqrt{u} - \frac{2}{5}u^{5/2} + C$$

Undoing the substitution(s):

$$= 2\sqrt{\sin(x)} - \frac{2}{5}(\sin(x))^{5/2} + C$$

Problem 59

$$\int x^3 e^{-x^2} dx$$

Rewriting the integral:

$$= \int x \cdot x^2 e^{-x^2} dx$$

Substituting

$$u = -x^2$$

$$-u = x^2$$

$$-du = 2xdx$$

yields:

$$= \frac{1}{2} \int ue^u du$$

Using integration by parts where

Sign	D	I
$+$	u	e^u
$-$	1	e^u
$+$	0	e^u

yields:

$$= \frac{1}{2} ue^u - \frac{1}{2} e^u + C$$

Undoing the substitution(s):

$$= -\frac{1}{2} x^2 e^{-x^2} - \frac{1}{2} e^{-x^2} + C$$

Problem 60

$$\int \sin\left(\sqrt{x}\right) dx$$

Substituting

$$u = \sqrt{x}$$

$$u^2 = x$$

$$2udu = dx$$

yields:

$$= 2 \int u \sin(u) du$$

Using integration by parts where

Sign	D	I
$+$	u	$\sin(u)$
$-$	1	$-\cos(u)$
$+$	0	$-\sin(u)$

yields:

$$= -2u\cos(u) + 2\sin(u) + C$$

Undoing the substitution(s):

$$= -2\sqrt{x}\cos\left(\sqrt{x}\right) + 2\sin\left(\sqrt{x}\right) + C$$

Problem 61

$$\int \frac{\arcsin(x)}{x^2} dx$$

Substituting

$$\sin\theta = x$$

$$\theta = \sin(x)$$

$$\cos\theta d\theta = dx$$

yields:

$$= \int \frac{\theta\cos\theta}{\sin^2\theta} d\theta$$

$$= \int u\cot\theta\csc\theta d\theta$$

Using integration by parts where

Sign	D	I
$+$	θ	$\cot\theta\csc\theta$
$-$	1	$-\csc\theta$
$+$	0	$\ln\lvert\csc\theta + \cot\theta\rvert$

yields:

$$= -\theta\csc\theta - \ln\left|\csc\theta + \cot\theta\right| + C$$

Undoing the substitution(s):

$$= -\frac{\arcsin(x)}{x} - \ln\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| + C$$

$$= -\frac{\arcsin(x)}{x} - \ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + C$$

Problem 62

$$\int \sqrt{x^2 - 9}dx$$

This integral is standard:

$$= \frac{1}{2}x\sqrt{x^2 - 9} - \frac{9}{2}\ln\left|\sqrt{x^2 - 9} + x\right| + C$$

Problem 63

$$\int x^2\sqrt{1 - x^2}dx$$

Substituting

$$\sin\theta = x$$

$$\cos\theta d\theta = dx$$

yields:

$$= \int \sin^2\theta\cos\theta\sqrt{1 - \sin^2\theta}d\theta$$

$$= \int \sin^2\theta\cos\theta\sqrt{\cos^2\theta}d\theta$$

$$= \int \sin^2\theta\cos^2\theta d\theta$$

$$= \frac{1}{4}\int 4\sin^2\theta\cos^2\theta d\theta$$

$$= \frac{1}{4}\int (2\sin\theta\cos\theta)^2 d\theta$$

$$= \frac{1}{4}\int \sin^2 2\theta d\theta$$

$$= \frac{1}{4}\int \left(\frac{1 - \cos 4\theta}{2}\right) d\theta$$

$$= \frac{1}{8}\int (1 - \cos 4\theta)d\theta$$

$$= \frac{1}{8}\int d\theta - \frac{1}{8}\int \cos 4\theta d\theta$$

Substituting

$$u = 4\theta$$

$$du = 4d\theta$$

yields:

$$= \frac{1}{8}\int d\theta - \frac{1}{32}\int \cos(u)du$$

Both integrals are standard:

$$= \frac{1}{8}\theta - \frac{1}{32}\sin(u) + C$$

Undoing the substitution(s):

$$= \frac{1}{8}\theta - \frac{1}{32}\sin 4\theta + C$$

$$= \frac{1}{8}\theta - \frac{1}{16}\sin 2\theta\cos 2\theta + C$$

$$= \frac{1}{8}\theta - \frac{1}{8}\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) + C$$

$$= \frac{1}{8}\arcsin(x) - \frac{1}{8}x\sqrt{1-x^2}\left(\left(\sqrt{1-x^2}\right)^2 - x^2\right) + C$$

$$= \frac{1}{8}\arcsin(x) - \frac{1}{8}x\sqrt{1-x^2}(1-x^2-x^2) + C$$

$$= \frac{1}{8}\arcsin(x) - \frac{1}{8}(x-2x^3)\sqrt{1-x^2} + C$$

Problem 64

$$\int x\sqrt{2x-x^2}dx$$

Rewriting the integral:

$$= \int \left(1 - \frac{1}{2}(2-2x)\right)\sqrt{2x-x^2}dx$$

$$= \int \sqrt{2x-x^2}dx - \frac{1}{2}\int (2-2x)\sqrt{2x-x^2}dx$$

$$= \int \sqrt{1-1+2x-x^2}dx - \frac{1}{2}\int (2-2x)\sqrt{2x-x^2}dx$$

$$= \int \sqrt{1-(1-2x+x^2)}dx - \frac{1}{2}\int (2-2x)\sqrt{2x-x^2}dx$$

$$= \int \sqrt{1-(1-x)^2}dx - \frac{1}{2}\int (2-2x)\sqrt{2x-x^2}dx$$

Substituting

$$u = 1-x \qquad v = 2x - x^2$$

$$du = -dx \qquad dv = (2-2x)dx$$

$$= -\int \sqrt{1-u^2}du - \frac{1}{2}\int \sqrt{v}dv$$

Both integrals are standard:

$$= -\frac{1}{2}\arcsin(u) - \frac{1}{2}u\sqrt{1-u^2} - \frac{1}{3}v^{3/2} + C$$

Undoing the substitution(s):

$$= -\frac{1}{2}\arcsin(1-x) - \frac{1-x}{2}\sqrt{2x-x^2} - \frac{1}{3}(2x-x^2)^{3/2} + C$$

Problem 65

$$\int \frac{x-2}{4x^2+4x+1}dx$$

Rewriting the integral:

$$= \int \frac{x-2}{(2x+1)^2}dx$$

Substituting

$$u = 2x+1$$

$$\frac{u-1}{2} = x$$

$$\frac{1}{2}du = dx$$

yields:

$$= \frac{1}{2}\int \frac{\frac{1}{2}u - \frac{5}{2}}{u^2} du$$

$$= \frac{1}{4}\int \frac{u-5}{u^2} du$$

$$= \frac{1}{4}\int \left(\frac{1}{u} - \frac{5}{u^2}\right) du$$

$$= \frac{1}{4}\int \frac{1}{u} du - \frac{5}{4}\int \frac{1}{u^2} du$$

Both integrals are standard:

$$= \frac{1}{4}\ln|u| + \frac{5}{4u} + C$$

Undoing the substitution(s):

$$= \frac{1}{4}\ln|2x+1| + \frac{5}{4(2x+1)} + C$$

$$= \frac{1}{4}\ln|2x+1| + \frac{5}{8x+4} + C$$

Problem 66

$$\int \frac{2x^2-5x-1}{x^3-2x^2-x+2} dx$$

Factoring the integral:

$$= \int \frac{2x^2-5x-1}{x^2(x-2)-(x-2)} dx$$

$$= \int \frac{2x^2-5x-1}{(x^2-1)(x-2)} dx$$

$$= \int \frac{2x^2-5x-1}{(x-1)(x+1)(x-2)} dx$$

Using partial fraction decomposition

$$= \int \left(\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2}\right) dx$$

where $A, B, C \in \mathbb{R}$ yields:

$$= \int \left(\frac{2}{x-1} + \frac{1}{x+1} + \frac{-1}{x-2} \right) dx$$

$$= 2\int \frac{1}{x-1}dx + \int \frac{1}{x+1}dx - \int \frac{1}{x-2}dx$$

Substituting

$$u_1 = x - 1 \qquad u_2 = x + 1 \qquad u_3 = x - 2$$

$$du_1 = dx \qquad du_2 = dx \qquad du_3 = dx$$

yields:

$$= 2\int \frac{1}{u_1}du_1 + \int \frac{1}{u_2}du_2 - \int \frac{1}{u_3}du_3$$

All three integrals are standard:

$$= 2\ln|u_1| + \ln|u_2| - \ln|u_3| + C$$

$$= \ln\left|u_1^2\right| + \ln|u_2| - \ln|u_3| + C$$

$$= \ln\left|\frac{u_1^2 u_2}{u_3}\right| + C$$

Undoing the substitution(s):

$$= \ln\left|\frac{(x-1)^2(x+1)}{x-2}\right| + C$$

Problem 67

$$\int \frac{e^{2x}}{e^{2x}-1}dx$$

Substituting

$$u = e^{2x} - 1$$

$$du = 2e^{2x}dx$$

yields:

$$= \frac{1}{2}\int \frac{1}{u}du$$

This integral is standard:

$$= \frac{1}{2}\ln|u| + C$$

Undoing the substitution(s):

$$= \frac{1}{2}\ln\left|e^{2x} - 1\right| + C$$

Problem 68

$$\int \frac{\cos(x)}{\sin^2(x) - 3\sin(x) + 2} dx$$

Substituting

$$u = \sin(x)$$

$$du = \cos(x) dx$$

yields:

$$= \int \frac{1}{u^2 - 3u + 2} du$$

Factoring the integral:

$$= \int \frac{1}{(u-2)(u-1)} du$$

Using partial fraction decomposition

$$= \int \left(\frac{A}{u-2} + \frac{B}{u-1} \right) du$$

where $A, B \in \mathbb{R}$ yields:

$$= \int \left(\frac{1}{u-2} + \frac{-1}{u-1} \right) du$$

$$= \int \frac{1}{u-2} du - \int \frac{1}{u-1} du$$

Substituting

$$v_1 = u - 2 \qquad v_2 = u - 1$$

$$dv_1 = du \qquad dv_2 = du$$

yields:

$$= \int \frac{1}{v_1} dv_1 - \int \frac{1}{v_2} dv_2$$

Both integrals are standard:

$$= \ln|v_1| - \ln|v_2| + C$$

$$= \ln\left|\frac{v_1}{v_2}\right| + C$$

Undoing the substitution(s):

$$= \ln\left|\frac{\sin(x) - 2}{\sin(x) - 1}\right| + C$$

Problem 69

$$\int \frac{2x^3+3x^2+4}{(x+1)^4}dx$$

Substituting

$$u = x+1$$

$$u-1 = x$$

$$du = dx$$

yields:

$$= \int \frac{2(u-1)^3+3(u-1)^2+4}{u^4}du$$

$$= \int \frac{2(u^3-3u^2+3u-1)+3(u^2-2u+1)+4}{u^4}du$$

$$= \int \frac{2u^3-6u^2+6u-2+3u^2-6u+3+4}{u^4}du$$

$$= \int \frac{2u^3-3u^2+5}{u^4}du$$

$$= \int \left(\frac{2}{u}-\frac{3}{u^2}+\frac{5}{u^4}\right)du$$

$$= 2\int \frac{1}{u}du - 3\int \frac{1}{u^2}du + 5\int \frac{1}{u^4}du$$

All three are standard integrals:

$$= 2\ln|u| + \frac{3}{u} - \frac{5}{3u^3} + C$$

Undoing the substitution(s):

$$= 2\ln|x+1| + \frac{3}{x+1} - \frac{5}{3(x+1)^3} + C$$

Problem 70

$$\int \frac{\sec^2(x)}{\tan^2(x)+2\tan(x)+2}dx$$

Rewriting the integral:

$$= \int \frac{\sec^2(x)}{\tan^2(x) + 2\tan(x) + 1 + 1} dx$$

$$= \int \frac{\sec^2(x)}{(\tan(x) + 1)^2 + 1} dx$$

Substituting

$$u = \tan(x) + 1$$

$$du = \sec^2(x) dx$$

yields:

$$= \int \frac{1}{u^2 + 1} du$$

This integral is standard:

$$= \arctan(u) + C$$

Undoing the substitution(s):

$$= \arctan(\tan(x) + 1) + C$$

Problem 71

$$\int \frac{x^3 + x^2 + 2x + 1}{x^4 + 2x^2 + 1} dx$$

Rewriting the integral:

$$= \int \frac{x^3 + x + x + x^2 + 1}{(x^2 + 1)^2} dx$$

$$= \int \frac{x + x(x^2 + 1) + x^2 + 1}{(x^2 + 1)^2} dx$$

$$= \int \left(\frac{x}{(x^2 + 1)^2} + \frac{x(x^2 + 1)}{(x^2 + 1)^2} + \frac{x^2 + 1}{(x^2 + 1)^2} \right) dx$$

$$= \int \frac{x}{(x^2 + 1)^2} dx + \int \frac{x}{x^2 + 1} dx + \int \frac{1}{x^2 + 1} dx$$

Substituting

$$u = x^2 + 1$$

$$du = 2x dx$$

yields:

$$= \frac{1}{2} \int \frac{1}{u^2} du + \frac{1}{2} \int \frac{1}{u} du$$

Both are standard integrals:

$$= -\frac{1}{2u} + \frac{1}{2} \ln |u| + \arctan(x) + C$$

Undoing the substitution(s):

$$= -\frac{1}{2x^2 + 2} + \frac{1}{2} \ln \left| x^2 + 1 \right| + \arctan(x) + C$$

Problem 72

$$\int \frac{3 + \cos(x)}{2 - \cos(x)} dx$$

Rewriting the integral:

$$= -\int \frac{\cos(x) + 3}{\cos(x) - 2} dx$$

$$= -\int \frac{\cos(x) - 2 + 5}{\cos(x) - 2} dx$$

$$= -\int \left(\frac{\cos(x) - 2}{\cos(x) - 2} + \frac{5}{\cos(x) - 2} \right) dx$$

$$= -\int dx - 5 \int \frac{1}{\cos(x) - 2} dx$$

Substituting

$$u = \tan\left(\frac{x}{2}\right)$$

$$du = \frac{1}{2} \sec^2\left(\frac{x}{2}\right) dx$$

$$\frac{2}{1 + u^2} du = dx$$

yields:

$$= -5 \int \frac{1}{\frac{1-u^2}{1+u^2} - 2} \cdot \frac{2}{1+u^2} du$$

$$= -10 \int \frac{1}{1 - u^2 - 2(1+u^2)} du$$

$$= -10 \int \frac{1}{1 - u^2 - 2 - 2u^2} du$$

$$= -10 \int \frac{1}{-1 - 3u^2} du$$

$$= 10 \int \frac{1}{1 + 3u^2} du$$

Both integrals are standard:

$$= -x + \frac{10}{\sqrt{3}} \arctan\left(\sqrt{3}u\right) + C$$

Undoing the substitution(s):

$$= -x + \frac{10}{\sqrt{3}} \arctan\left(\sqrt{3}\tan\left(\frac{x}{2}\right)\right) + C$$

Problem 73

$$\int x^5 \sqrt{x^3 - 1} dx$$

Rewriting the integral:

$$= \int x^2 \cdot x^3 \sqrt{x^3 - 1} dx$$

Substituting

$$u = x^3 - 1$$

$$u + 1 = x^3$$

$$du = 3x^2 dx$$

yields:

$$= \frac{1}{3} \int (u+1)\sqrt{u} du$$

$$= \frac{1}{3} \int \left(u^{3/2} + \sqrt{u}\right) du$$

$$= \frac{1}{3} \int u^{3/2} du + \frac{1}{3} \int \sqrt{u} du$$

Both integrals are standard:

$$= \frac{2}{15}u^{5/2} + \frac{2}{9}u^{3/2} + C$$

Undoing the substitution(s):

$$= \frac{2}{15}(x^3 - 1)^{5/2} + \frac{2}{9}(x^3 - 1)^{3/2} + C$$

Problem 74

$$\int \frac{1}{2 + 2\cos(x) + \sin(x)} dx$$

Substituting

$$u = \tan\left(\frac{x}{2}\right)$$

$$du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx$$

$$\frac{2}{1+u^2} du = dx$$

yields:

$$= \int \frac{1}{2 + 2\left(\frac{1-u^2}{1+u^2}\right) + \frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2} du$$

$$= 2\int \frac{1}{2(1+u^2) + 2(1-u^2) + 2u} du$$

$$= 2\int \frac{1}{2 + 2u^2 + 2 - 2u^2 + 2u} du$$

$$= 2\int \frac{1}{4 + 2u} du$$

$$= \int \frac{1}{2 + u} du$$

Substituting

$$v = 2 + u$$

$$dv = du$$

yields:

$$= \int \frac{1}{v} dv$$

This integral is standard:

$$= \ln|v| + C$$

Undoing the substitution(s):

$$= \ln\left|2 + \tan\left(\frac{x}{2}\right)\right| + C$$

Problem 75

$$\int \frac{\sqrt{1+\sin(x)}}{\sec(x)} dx$$

Rewriting the integral:

$$= \int \cos(x)\sqrt{1+\sin(x)} dx$$

Substituting

$$u = 1 + \sin(x)$$

$$du = \cos(x) dx$$

yields:

$$= \int \sqrt{u} du$$

This integral is standard:

$$= \frac{2}{3} u^{3/2} + C$$

Undoing the substitution(s):

$$= \frac{2}{3}(1 + \sin(x))^{3/2} + C$$

Problem 76

$$\int \frac{1}{x^{2/3}\left(1 + x^{2/3}\right)} dx$$

Substituting

$$u = \sqrt[3]{x}$$

$$du = \frac{1}{3x^{2/3}} dx$$

yields:

$$= 3 \int \frac{1}{1+u^2} du$$

This integral is standard:

$$= 3 \arctan(u) + C$$

Undoing the substitution(s):

$$= 3 \arctan\left(\sqrt[3]{x}\right) + C$$

Problem 77

$$\int \frac{\sin(x)}{\sin(2x)} dx$$

Rewriting the integral:

$$= \int \frac{\sin(x)}{2\sin(x)\cos(x)} dx$$

$$= \frac{1}{2} \int \frac{1}{\cos(x)} dx$$

$$= \frac{1}{2} \int \sec(x) dx$$

This integral is standard:

$$= \frac{1}{2} \ln|\sec(x) + \tan(x)| + C$$

Problem 78

$$\int \sqrt{1 + \cos(x)} dx$$

Rewriting the integral:

$$= \int \sqrt{1 + \cos(x)} \cdot \frac{\sqrt{1 - \cos(x)}}{\sqrt{1 - \cos(x)}} dx$$

$$= \int \frac{\sqrt{1 - \cos^2(x)}}{\sqrt{1 - \cos(x)}} dx$$

$$= \int \frac{\sqrt{\sin^2(x)}}{\sqrt{1 - \cos(x)}} dx$$

$$= \int \frac{\sin(x)}{\sqrt{1 - \cos(x)}} dx$$

Substituting

$$u = 1 - \cos(x)$$

$$du = \sin(x) dx$$

yields:

$$= \int \frac{1}{\sqrt{u}} du$$

This integral is standard:

$$= 2\sqrt{u} + C$$

Undoing the substitution(s):

$$= 2\sqrt{1 - \cos(x)} + C$$

Problem 79

$$\int \sqrt{1 + \sin(x)} dx$$

Rewriting the integral:

$$= \int \sqrt{1 + \sin(x)} \cdot \frac{\sqrt{1 - \sin(x)}}{\sqrt{1 - \sin(x)}} dx$$

$$= \int \frac{1 - \sin^2(x)}{\sqrt{1 - \sin(x)}} dx$$

$$= \int \frac{\sqrt{\cos^2(x)}}{\sqrt{1 - \sin(x)}} dx$$

$$= \int \frac{\cos(x)}{\sqrt{1 - \sin(x)}} dx$$

Substituting

$$u = 1 - \sin(x)$$

$$du = -\cos(x) dx$$

yields:

$$= -\int \frac{1}{\sqrt{u}} du$$

This integral is standard:

$$= -2\sqrt{u} + C$$

Problem 80

$$\int \frac{\sec^2(x)}{1 - \tan^2(x)} dx$$

Substituting

$$u = \tan(x)$$

$$du = \sec^2(x) dx$$

yields:

$$= \int \frac{1}{1-u^2} du$$

This integral is standard:

$$= \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + C$$

Undoing the substitution(s):

$$= \frac{1}{2} \ln \left| \frac{1+\tan(x)}{1-\tan(x)} \right| + C$$

Problem 81

$$\int \ln\left(x^2+x+1\right) dx$$

Using integration by parts where

$$u = \ln(x^2+x+1) \qquad v = x$$

$$du = \frac{2x+1}{x^2+x+1} dx \qquad dv = dx$$

yields:

$$= x\ln(x^2+x+1) - \int \frac{2x^2+x}{x^2+x+1} dx$$

Performing long division on the integral yields:

$$= -\int \left(2 + \frac{-x-2}{x^2+x+1}\right) dx$$

$$= -2\int dx + \int \frac{x+2}{x^2+x+1} dx$$

$$= -2\int dx + \int \frac{x+\frac{1}{2}+\frac{3}{2}}{x^2+x+1} dx$$

$$= -2\int dx + \frac{1}{2}\int \frac{2x+1}{x^2+x+1} dx + \frac{3}{2}\int \frac{1}{x^2+x+1} dx$$

$$= -2\int dx + \frac{1}{2}\int \frac{2x+1}{x^2+x+1} dx + \frac{3}{2}\int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}} dx$$

$$= -2\int dx + \frac{1}{2}\int \frac{2x+1}{x^2+x+1} dx + 6\int \frac{1}{4x^2+4x+1+3} dx$$

$$= -2\int dx + \frac{1}{2}\int \frac{2x+1}{x^2+x+1} dx + 6\int \frac{1}{(2x+1)^2+3} dx$$

Substituting

$$w = x^2 + x + 1 \qquad\qquad z = 2x + 1$$

$$dw = (2x+1)dx \qquad\qquad dz = 2dx$$

yields:

$$= -2\int dx + \frac{1}{2}\int \frac{1}{w}dw + 3\int \frac{1}{z^2+3}$$

All three integrals are standard:

$$= -2x + \frac{1}{2}\ln|w| + \frac{3}{\sqrt{3}}\arctan\left(\frac{z}{\sqrt{3}}\right) + C$$

Undoing the substitution(s):

$$= x\ln(x^2+x+1) - 2x + \frac{1}{2}\ln\left|x^2+x+1\right| + \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Problem 82

$$\int e^x \arcsin(e^x)\, dx$$

Substituting

$$u = e^x$$

$$du = e^x dx$$

yields:

$$= \int \arcsin(u) du$$

This integral is standard:

$$= u\arcsin(u) + \sqrt{1-u^2} + C$$

Undoing the substitution(s):

$$= e^x \arcsin(e^x) + \sqrt{1-e^{2x}} + C$$

Problem 83

$$\int \frac{\arctan(x)}{x^2}dx$$

Using integration by parts where:

$$u = \arctan(x) \qquad\qquad v = -\frac{1}{x}$$

$$du = \frac{1}{1+x^2}dx \qquad\qquad dv = \frac{1}{x^2}dx$$

yields:

$$= -\frac{\arctan(x)}{x} + \int \frac{1}{x(1+x^2)} dx$$

Using partial fraction decomposition

$$= \int \left(\frac{A}{x} + \frac{Bx + C}{1 + x^2} \right) dx$$

where $A, B, C \in \mathbb{R}$ yields:

$$= \int \left(\frac{1}{x} - \frac{x}{x^2 + 1} \right) dx$$

$$= \int \frac{1}{x} dx - \int \frac{x}{x^2 + 1} dx$$

Substituting

$$u = x^2 + 1$$

$$du = 2x dx$$

yields:

$$= -\frac{1}{2} \int \frac{1}{u} du$$

This integral is standard:

$$= -\frac{1}{2} \ln |u| + C$$

Undoing the substitution(s):

$$= -\frac{\arctan(x)}{x} + \ln |x| - \frac{1}{2} \ln \left| x^2 + 1 \right| + C$$

$$= -\frac{\arctan(x)}{x} + \ln \left| \frac{x}{\sqrt{x^2 + 1}} \right| + C$$

Problem 84

$$\int \frac{x^2}{\sqrt{x^2 - 25}} dx$$

Rewriting the integral:

$$= \int \frac{x^2}{\sqrt{25 \left(\frac{x^2}{25} - 1 \right)}} dx$$

$$= \frac{1}{5} \int \frac{x^2}{\sqrt{\left(\frac{x}{5} \right)^2 - 1}} dx$$

Substituting

$$\sec\theta = \frac{x}{5}$$

$$5\sec\theta = x$$

$$5\sec\theta\tan\theta d\theta = dx$$

yields:

$$= \int \frac{(5\sec\theta)^2}{\sqrt{\sec^2\theta - 1}} \sec\theta\tan\theta d\theta$$

$$= 25\int \frac{\sec^3\theta\tan\theta}{\sqrt{\tan^2\theta}} d\theta$$

$$= 25\int \sec^3\theta d\theta$$

This integral is standard:

$$= 25\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C\right)$$

$$= \frac{25}{2}\sec\theta\tan\theta + \frac{25}{2}\ln|\sec\theta + \tan\theta| + C$$

Undoing the substitution(s):

$$= \frac{25}{2}\cdot\frac{x}{5}\cdot\frac{\sqrt{x^2-25}}{5} + \frac{25}{2}\ln\left|\frac{x}{5} + \frac{\sqrt{x^2-25}}{5}\right| + C$$

$$= \frac{1}{2}x\sqrt{x^2-25} + \frac{25}{2}\ln\left|x + \sqrt{x^2-25}\right| + C$$

Problem 85

$$\int \frac{x^3}{\left(x^2+1\right)^2} dx$$

Rewriting the integral:

$$= \int \frac{x \cdot x^2}{(x^2+1)^2} dx$$

Substituting

$$u = x^2 + 1$$

$$u - 1 = x^2$$

$$du = 2xdx$$

yields:

$$= \frac{1}{2} \int \frac{u-1}{u^2} du$$

$$= \frac{1}{2} \int \left(\frac{1}{u} - \frac{1}{u^2} \right) du$$

$$= \frac{1}{2} \int \frac{1}{u} du - \frac{1}{2} \int \frac{1}{u^2} du$$

All three integrals are standard:

$$= \frac{1}{2} \ln |u| - \frac{1}{2u} + C$$

Undoing the substitution(s):

$$= \frac{1}{2} \ln \left| x^2 + 1 \right| + \frac{1}{2x^2 + 2} + C$$

Problem 86

$$\int \frac{1}{x\sqrt{6x - x^2}} dx$$

Rewriting the integral:

$$= \int \frac{1}{x\sqrt{9 - 9 + 6x - x^2}} dx$$

$$= \int \frac{1}{x\sqrt{9 - (9 - 6x + x^2)}} dx$$

$$= \int \frac{1}{x\sqrt{9 - (3 - x)^2}} dx$$

$$= \int \frac{1}{x\sqrt{9 \left(1 - \frac{(3-x)^2}{9}\right)}} dx$$

$$= \frac{1}{3} \int \frac{1}{x\sqrt{1 - \left(\frac{3-x}{3}\right)^2}} dx$$

Substituting

$$\sin\theta = \frac{3-x}{3}$$

$$3 - 3\sin\theta = x$$

$$-3\cos\theta d\theta = dx$$

yields:

$$= -\int \frac{\cos\theta}{(3-3\sin\theta)\sqrt{1-\sin^2\theta}}d\theta$$

$$= -\int \frac{\cos\theta}{(3-3\sin\theta)\sqrt{\cos^2\theta}}d\theta$$

$$= -\int \frac{1}{3-3\sin\theta}d\theta$$

$$= -\frac{1}{3}\int \frac{1}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}d\theta$$

$$= -\frac{1}{3}\int \frac{1+\sin\theta}{1-\sin^2\theta}d\theta$$

$$= -\frac{1}{3}\int \frac{1+\sin\theta}{\cos^2\theta}d\theta$$

$$= -\frac{1}{3}\int \left(\sec^2\theta + \sec\theta\tan\theta\right)d\theta$$

$$= -\frac{1}{3}\int \sec^2\theta d\theta - \frac{1}{3}\int \sec\theta\tan\theta d\theta$$

Both integrals are standard:

$$= -\frac{1}{3}\tan\theta - \frac{1}{3}\sec\theta + C$$

Undoing the substitution(s):

$$= -\frac{1}{3}\cdot\frac{3-x}{\sqrt{6x-x^2}} - \frac{1}{3}\cdot\frac{3}{\sqrt{6x-x^2}} + C$$

$$= \frac{x-3}{3\sqrt{6x-x^2}} - \frac{3}{3\sqrt{6x-x^2}} + C$$

$$= \frac{x-6}{3\sqrt{6x-x^2}} + C$$

Problem 87

$$\int \frac{3x+2}{(x^2+4)^{3/2}}dx$$

Rewriting the integral:

$$\frac{3}{2}\int \frac{2x}{(x^2+4)^{3/2}}dx + \frac{1}{4}\int \frac{1}{\left(\left(\frac{x}{2}\right)^2+1\right)^{3/2}}dx$$

Substituting

$$u = x^2 + 4 \qquad \tan\theta = \frac{x}{2}$$

$$du = 2xdx \qquad \sec^2\theta d\theta = \frac{1}{2}dx$$

yields:

$$= \frac{3}{2}\int \frac{1}{u^{3/2}}du + \frac{1}{2}\int \frac{\sec^2\theta}{(\tan^2\theta + 1)^{3/2}}d\theta$$

$$= \frac{3}{2}\int \frac{1}{u^{3/2}}du + \frac{1}{2}\int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}d\theta$$

$$= \frac{3}{2}\int \frac{1}{u^{3/2}}du + \frac{1}{2}\int \frac{1}{\sec\theta}d\theta$$

$$= \frac{3}{2}\int \frac{1}{u^{3/2}}du + \frac{1}{2}\int \cos\theta d\theta$$

Both integrals are standard:

$$= -\frac{3}{\sqrt{u}} + \frac{1}{2}\sin\theta + C$$

Undoing the substitution(s):

$$= -\frac{3}{\sqrt{x^2+4}} + \frac{1}{2}\cdot\frac{x}{\sqrt{x^2+4}} + C$$

$$= \frac{x-6}{2\sqrt{x^2+4}} + C$$

Problem 88

$$\int x^{3/2}\ln(x)\,dx$$

Using integration by parts where

$$u = \ln(x) \qquad v = \frac{2}{5}x^{5/2}$$

$$du = \frac{1}{x}dx \qquad dv = x^{3/2}dx$$

yields:

$$= \frac{2}{5}x^{5/2}\ln(x) - \frac{2}{5}\int x^{3/2}dx$$

This integral is standard:

$$= \frac{2}{5}x^{5/2}\ln(x) - \frac{4}{25}x^{5/2} + C$$

Problem 89

$$\int \frac{\sqrt{1+\sin^2(x)}}{\sec(x)\csc(x)}dx$$

Rewriting the integral;

$$= \int \sin(x)\cos(x)\sqrt{1+\sin^2(x)}dx$$

Substituting

$$u = 1+\sin^2(x)$$

$$du = 2\sin(x)\cos(x)dx$$

yields:

$$= \frac{1}{2}\int \sqrt{u}du$$

This integral is standard:

$$= \frac{1}{3}u^{3/2} + C$$

Undoing the substitution(s):

$$= \frac{1}{3}\left(1+\sin^2(x)\right)^{3/2} + C$$

Problem 90

$$\int \frac{e^{\sqrt{\sin(x)}}}{\sec(x)\sqrt{\sin(x)}}dx$$

Rewriting the integral:

$$= \int \frac{\cos(x)e^{\sqrt{\sin(x)}}}{\sqrt{\sin(x)}}dx$$

Substituting

$$u = \sqrt{\sin(x)}$$

$$du = \frac{\cos(x)}{2\sqrt{\sin(x)}}dx$$

yields

$$= 2\int e^u du$$

This integral is standard:

$$= 2e^u + C$$

Undoing the substitution(s):

$$= 2e^{\sqrt{\sin(x)}} + C$$

Problem 91

$$\int xe^x \sin(x)dx$$

Recall that

$$\int e^x \sin(x)dx = \frac{1}{2}e^x \sin(x) - \frac{1}{2}e^x \cos(x)$$

Using integration by parts where

Sign	D	I
$+$	xe^x	$\sin(x)$
$-$	$e^x + xe^x$	$-\cos(x)$
$+$	$2e^x + xe^x$	$-\sin(x)$

yields:

$$= -xe^x \cos(x) + e^x \sin(x)(1+x) - \int e^x \sin(x)(2+x)dx$$

$$2\int xe^x \sin(x)dx = -xe^x \cos(x) + e^x \sin(x)(1+x) - 2\int e^x \sin(x)dx$$

$$= -xe^x \cos(x) + xe^x \sin(x) + e^x \sin(x) - e^x \sin(x) + e^x \cos(x)$$

$$\int xe^x \sin(x)dx = \frac{1}{2}xe^x(\sin(x) - \cos(x)) + \frac{1}{2}e^x \cos(x) + C$$

Problem 92

$$\int x^2 e^{x^{3/2}} dx$$

Rewriting the integral:

$$= \int x^{3/2} \cdot x^{1/2} e^{x^{3/2}} dx$$

Substituting

$$u = x^{3/2}$$

$$du = \frac{3}{2}x^{1/2}dx$$

yields:

$$= \frac{2}{3}\int ue^u du$$

Using integration by parts where

$$w = u \qquad v = e^u$$

$$dw = du \qquad dv = e^u du$$

yields:

$$= \frac{2}{3}ue^u - \frac{2}{3}\int e^u du$$

This integral is standard:

$$= \frac{2}{3}ue^u - \frac{2}{3}e^u + C$$

Undoing the substitution(s):

$$= \frac{2}{3}x^{3/2}e^{x^{3/2}} - \frac{2}{3}e^{x^{3/2}} + C$$

Problem 93

$$\int \frac{\arctan(x)}{(x-1)^3}dx$$

Using integration by parts where

$$u = \arctan(x) \qquad v = -\frac{1}{2(x-1)^2}dx$$

$$du = \frac{1}{1+x^2}dx \qquad dv = \frac{1}{(x-1)^3}dx$$

yields:

$$= -\frac{\arctan(x)}{2(x-1)^2} + \frac{1}{2}\int \frac{1}{(1+x^2)(x-1)^2}dx$$

Using partial fraction decomposition

$$= \frac{1}{2}\int \left(\frac{Ax+B}{1+x^2} + \frac{C}{(x-1)^2} + \frac{D}{x-1}\right)dx$$

where $A, B, C, D \in \mathbb{R}$ yields:

$$= \frac{1}{2}\int \left(\frac{\frac{1}{2}x+0}{x^2+1} + \frac{-\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{(x-1)^2}\right)dx$$

$$= \frac{1}{4}\int \left(\frac{x}{x^2+1} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\right)dx$$

$$= \frac{1}{4}\int \frac{x}{x^2+1}dx - \frac{1}{4}\int \frac{1}{x-1}dx + \frac{1}{4}\int \frac{1}{(x-1)^2}dx$$

Substituting

$$w_1 = x^2+1 \qquad w_2 = x-1$$

$$dw_1 = 2xdx \qquad dw_2 = dx$$

yields:

$$= \frac{1}{8}\int \frac{1}{w_1}dw_1 - \frac{1}{4}\int \frac{1}{w_2}dw_2 + \frac{1}{4}\int \frac{1}{w_2^2}dw_2$$

All three integrals are standard:

$$= \frac{1}{8}\ln|w_1| - \frac{1}{4}\ln|w_2| - \frac{1}{4w_2}$$

$$= \frac{1}{8}\ln\left|\frac{w_1}{w_2^2}\right| - \frac{1}{4w_2}$$

Undoing the substitution(s):

$$= -\frac{\arctan(x)}{2(x-1)^2} + \frac{1}{8}\ln\left|\frac{x^2+1}{(x-1)^2}\right| - \frac{1}{4(x-1)} + C$$

Problem 94

$$\int \ln\left(1+\sqrt{x}\right)dx$$

Substituting

$$u = 1 + \sqrt{x}$$

$$u - 1 = \sqrt{x}$$

$$u^2 - 2u + 1 = x$$

$$(2u-2)du = dx$$

yields:

$$= \int \ln(u)(2u-2)du$$

$$= 2\int u\ln(u)du - 2\int \ln(u)du$$

Using integration by parts where

$$w = \ln(u) \qquad v = \frac{1}{2}u^2$$

$$dw = \frac{1}{u}du \qquad dv = udu$$

yields:

$$= 2\left(\frac{1}{2}u^2\ln(u) - \frac{1}{2}\int udu\right)$$

$$= u^2\ln(u) - \int udu$$

The remaining two integrals are standard:

$$= u^2\ln(u) - \frac{1}{2}u^2 - 2u\ln(u) + 2u + C$$

$$= u\ln(u)\,(u-2) - \frac{1}{2}u^2 + 2u + C$$

Undoing the substitution(s):

$$= \left(1+\sqrt{x}\right)\ln\left(1+\sqrt{x}\right)\left(\sqrt{x}-1\right) - \frac{1}{2}\left(1+\sqrt{x}\right)^2 + 2\sqrt{x} + C$$

Problem 95

$$\int \frac{2x+3}{\sqrt{3+6x-9x^2}}dx$$

Rewriting the integral:

$$= \int \frac{2x - \frac{2}{3} + \frac{11}{3}}{\sqrt{3+6x-9x^2}}dx$$

$$= -\frac{1}{9}\int \frac{6-18x}{\sqrt{3+6x-9x^2}} + \frac{11}{3}\int \frac{1}{\sqrt{4-1+6x-9x^2}}dx$$

$$= -\frac{1}{9}\int \frac{6-18x}{\sqrt{3+6x-9x^2}} + \frac{11}{3}\int \frac{1}{\sqrt{4-(1-6x+9x^2)}}dx$$

$$= -\frac{1}{9}\int \frac{6-18x}{\sqrt{3+6x-9x^2}} + \frac{11}{3}\int \frac{1}{\sqrt{4-(1-3x)^2}}dx$$

Substituting

$$u = 3+6x-9x^2 \qquad v = 1-3x$$

$$du = (6-18x)dx \qquad dv = -3dx$$

yields:

$$= -\frac{1}{9}\int \frac{1}{\sqrt{u}}du - \frac{11}{9}\int \frac{1}{\sqrt{4-v^2}}dv$$

Both integrals are standard:

$$= -\frac{2}{9}\sqrt{u} - \frac{11}{9}\arcsin\left(\frac{v}{2}\right) + C$$

Undoing the substitution(s):

$$= -\frac{2}{9}\sqrt{3+6x-9x^2} - \frac{11}{9}\arcsin\left(\frac{1-3x}{2}\right) + C$$

Problem 96

$$\int \frac{1}{2+2\sin(x)+\cos(x)}dx$$

Substituting

$$u = \tan\left(\frac{x}{2}\right)$$

$$du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$$

$$\frac{2}{1+u^2}du = dx$$

yields:

$$= \int \frac{1}{2+2\left(\frac{2u}{1+u^2}\right)+\left(\frac{1-u^2}{1+u^2}\right)} \cdot \frac{2}{1+u^2}du$$

$$= 2\int \frac{1}{2+2u^2+4u+1-u^2}du$$

$$= 2\int \frac{1}{u^2+4u+3}du$$

$$= 2\int \frac{1}{(u+3)(u+1)}du$$

$$= 2\int \left(\frac{-\frac{1}{2}}{u+3} + \frac{\frac{1}{2}}{u+1}\right)du$$

$$= -\int \frac{1}{u+3}du + \int \frac{1}{u+1}du$$

Substituting

$$v_1 = u + 3 \qquad\qquad v_2 = u + 1$$

$$dv_1 = du \qquad\qquad dv_2 = du$$

yields:

$$= -\int \frac{1}{v_1} dv_1 + \int \frac{1}{v_2} dv_2$$

Both integrals are standard:

$$= -\ln|v_1| + \ln|v_2| + C$$

$$= \ln\left|\frac{v_2}{v_1}\right| + C$$

Undoing the substitution(s):

$$\ln\left|\frac{\tan\left(\frac{x}{2}\right) + 1}{\tan\left(\frac{x}{2}\right) + 3}\right| + C$$

Problem 97

$$\int \frac{\sin^3(x)}{\cos(x) - 1} dx$$

Rewriting the integral:

$$= -\int \frac{\sin(x) \cdot \sin^2(x)}{1 - \cos(x)} dx$$

$$= -\int \frac{\sin(x)(1 - \cos^2(x))}{1 - \cos(x)} dx$$

$$= -\int \frac{\sin(x)(1 - \cos(x))(1 + \cos(x))}{1 - \cos(x)} dx$$

$$= -\int \sin(x)(1 + \cos(x)) dx$$

Substituting

$$u = 1 + \cos(x)$$

$$du = -\sin(x) dx$$

yields:

$$= \int u du$$

This integral is standard:

$$= \frac{1}{2}u^2 + C$$

Undoing the substitution(s):

$$= \frac{1}{2}(1 + \cos(x))^2 + C$$

Problem 98

$$\int x^{3/2} \arctan\left(\sqrt{x}\right) dx$$

Using integration by parts where:

$$u = \arctan\left(\sqrt{x}\right) \qquad v = \frac{2}{5}x^{5/2}$$

$$dw = \frac{1}{2\sqrt{x}(1+x)}dx \qquad dv = x^{3/2}dx$$

yields:

$$= \frac{2}{5}x^{5/2} \arctan\left(\sqrt{x}\right) - \frac{1}{5}\int \frac{x^2}{1+x}dx$$

Substituting

$$w = 1 + x$$

$$w - 1 = x$$

$$dw = dx$$

yields:

$$= -\frac{1}{5}\int \frac{(w-1)^2}{w}dw$$

$$= -\frac{1}{5}\int \frac{w^2 - 2w + 1}{w}dw$$

$$= -\frac{1}{5}\int \left(w - 2 + \frac{1}{w}\right) dw$$

$$= -\frac{1}{5}\int wdw + \frac{2}{5}\int dw - \frac{1}{5}\int \frac{1}{w}dw$$

All three integrals are standard:

$$-\frac{1}{10}w^2 + \frac{2}{5}w - \frac{1}{5}\ln|w| + C$$

Undoing the substitution(s):

$$= \frac{2}{5}x^{5/2} \arctan\left(\sqrt{x}\right) - \frac{1}{10}(1+x)^2 + \frac{2}{5}(1+x) - \frac{1}{5}\ln|1+x| + C$$

Problem 99

$$\int \operatorname{arcsec}\left(\sqrt{x}\right) dx$$

Substituting

$$u = \sqrt{x}$$

$$u^2 = x$$

$$2udu = dx$$

yields:

$$= 2\int u \operatorname{arcsec}(u) du$$

Using integration by parts where:

$$w = \operatorname{arcsec}(u) \qquad v = \frac{1}{2}u^2$$

$$dw = \frac{1}{u\sqrt{u^2-1}}du \qquad dv = udu$$

yields:

$$= 2\left(\frac{1}{2}u^2 \operatorname{arcsec}(u) - \frac{1}{2}\int \frac{u^2}{u\sqrt{u^2-1}}du\right)$$

$$= u^2 \operatorname{arcsec}(u) - \int \frac{u}{\sqrt{u^2-1}}du$$

Substituting

$$z = u^2 - 1$$

$$dz = 2udu$$

yields:

$$= u^2 \operatorname{arcsec}(u) - \frac{1}{2}\int \frac{1}{\sqrt{z}}dz$$

This integral is standard:

$$= u^2 \operatorname{arcsec}(u) - \sqrt{z} + C$$

Undoing the substitution(s):

$$= x \operatorname{arcsec}\left(\sqrt{x}\right) - \sqrt{x-1} + C$$

Problem 100

$$\int x\sqrt{\frac{1-x^2}{1+x^2}}dx$$

Substituting

$$\sin\theta = x^2$$

$$\cos\theta d\theta = 2xdx$$

yields:

$$= \frac{1}{2}\int \cos\theta\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}d\theta$$

$$= \frac{1}{2}\int \cos\theta\sqrt{\frac{1-\sin\theta}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta}}d\theta$$

$$= \frac{1}{2}\int \cos\theta\sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}d\theta$$

$$= \frac{1}{2}\int \cos\theta\sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}d\theta$$

$$= \frac{1}{2}\int \cos\theta\cdot\frac{1-\sin\theta}{\cos\theta}d\theta$$

$$= \frac{1}{2}\int (1-\sin\theta)d\theta$$

$$= \frac{1}{2}\int d\theta - \frac{1}{2}\int \sin\theta d\theta$$

Both integrals are standard:

$$= \frac{1}{2}\theta + \frac{1}{2}\cos\theta + C$$

Undoing the substitution(s):

$$= \frac{1}{2}\arcsin\left(x^2\right) + \frac{1}{2}\sqrt{1-x^4} + C$$

4 Advanced Integrals - Proofs

Advanced Integral 1

$$\int \sec^3(x)dx$$

Rewriting the integral:

$$= \int \sec(x)\sec^2(x)dx$$

Using integration by parts where

$$u = \sec(x) \qquad v = \tan(x)$$

$$du = \sec(x)\tan(x)dx \qquad dv = \sec^2(x)dx$$

yields:

$$\begin{aligned}
&= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)dx \\
&= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x) - 1)dx \\
&= \sec(x)\tan(x) - \int (\sec^3(x) - \sec(x))dx \\
&= \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx \\
2\int \sec^3(x)dx &= \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C \\
&= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x)| + C
\end{aligned}$$

Advanced Integral 2

$$\int \arcsin(x)dx$$

Using integration by parts where

$$u = \arcsin(x) \qquad v = x$$

$$du = \frac{1}{\sqrt{1 - x^2}}dx \qquad dv = dx$$

yields:

$$x\arcsin(x) - \int \frac{x}{\sqrt{1 - x^2}}dx$$

Substituting

$$u = 1 - x^2$$

$$du = -2xdx$$

yields:

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

This integral is standard:

$$= \sqrt{u} + C$$

Undoing the substitution(s):

$$= x \arcsin(x) + \sqrt{1 - x^2} + C$$

Advanced Integral 3

$$\int \arccos(x) dx$$

Using integration by parts where

$$u = \arccos(x) \qquad v = x$$

$$du = -\frac{1}{\sqrt{1 - x^2}} dx \qquad dv = dx$$

yields:

$$x \arccos(x) + \int \frac{x}{\sqrt{1 - x^2}} dx$$

Substituting

$$u = 1 - x^2$$

$$du = -2xdx$$

yields:

$$= -\frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

This integral is standard:

$$= -\sqrt{u} + C$$

Undoing the substitution(s):

$$= x \arccos(x) - \sqrt{1 - x^2} + C$$

Advanced Integral 4

$$\int \arctan(x)dx$$

Using integration by parts where

$$u = \arctan(x) \qquad v = x$$

$$du = \frac{1}{1+x^2}dx \qquad dv = dx$$

yields:

$$= x\arctan(x) - \int \frac{x}{1+x^2}dx$$

Substituting

$$u = 1 + x^2$$

$$du = 2xdx$$

yields:

$$= -\frac{1}{2}\int \frac{1}{u}du$$

This integral is standard:

$$= x\arctan(x) - \frac{1}{2}\ln\left|1+x^2\right| + C$$

Advanced Integral 5

$$\int \sqrt{x^2+a^2}dx$$

Rewriting the integral:

$$= \int \sqrt{a^2\left(\frac{x^2}{a^2}+1\right)}dx$$

$$= a\int \sqrt{\left(\frac{x}{a}\right)^2+1}dx$$

Substituting

Advanced Integral 6

$$\int \sqrt{x^2-a^2}dx$$

Advanced Integral 7

$$\int \sqrt{a^2 - x^2} dx$$

Advanced Integral 8

$$\int \frac{1}{x^2 + a^2} dx$$

Advanced Integral 9

$$\int \frac{1}{x^2 - a^2} dx$$

Advanced Integral 10

$$\int \frac{1}{a^2 - x^2} dx$$

Advanced Integral 11

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx$$

Advanced Integral 12

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx$$

Advanced Integral 13

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx$$

Advanced Integral 14

$$\int \sin^n(x) dx$$

Advanced Integral 15

$$\int \cos^n(x) dx$$